Root and Ratio tests, Absolute Convergence

Last time we saw that it is much easier for an alternating series to converge than it is for a series of positive terms. The alternating series test tells us that if a_n is positive and decreasing and approaches zero then $\sum(-1)^n a_n$ is convergent.

It is often important to be able to decide if a series converges because of "cancellation" between positive and negative terms. This leads to the following concept:

Definition: Absolute Convergence A series

$$\sum a_n$$

is absolutely convergent if the series

$$\sum |a_n|$$

is also convergent.

A series which is not absolutely convergent (in other words a series for which

$$\sum a_n$$

converges but for which

$$\sum |a_n|$$

diverges is said to be be "conditionally convergent".

Example 1a:

$$\sum(-1)^n \frac{n^2}{(n+1)^3}$$

Example 1b:

$$\sum \frac{(-1)^n}{n^3}$$

We are going to learn two new convergence tests: the ratio test and the root test. Let me first state the two tests:

Theorem: Ratio Test Consider the series

$$\sum a_n$$

and assume that the limit of the ratio exists:

$$R = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

- If $R < 1$ then the series $\sum a_n$ converges absolutely.
- If $R > 1$ then the series diverges.
- If $R = 1$ the test fails.

Note: The book additionally notes that if $R = \infty$ then the series diverges, but this is a little bit sloppy and needs to be interpreted carefully: what this **really** means is that "all terms" go to infinity. More precisely for every $L > 0$ there exists an N such that for all $n > N$ we have

$$|a_n| > L$$

We'll give an example of how being a bit sloppy about the ratio "going to infinity" can get you into trouble.

Note: It is not necessary that the limit exist. If there exists an N such that for all $n > N$ we have

$$\frac{|a_{n+1}|}{|a_n|} < R < 1.$$

then the series converges. We'll give an example of this later.

Application Note: The ratio test is a theorem and thus applies to all series, but it tends to be most effective for expressions involving exponentials, factorials, and possibly lower order terms. Factorials and powers have the property that the $n+1^{st}$ term can be expressed neatly in terms of the n^{th} term.

The next important theorem is the root test. It is very similar to the ratio test.

Theorem: Root Test: Suppose that for the series

$$\sum a_n$$

the following limit exists

$$\lim |a_n|^{\frac{1}{n}} = R$$

Then

- If $R < 1$ the series is absolutely convergent.
- If $R > 1$ the series is divergent.
- If $R = 1$ the test fails.

Note: It is not necessary for the limit to actually exist. It is sufficient that for all $n > N$ we have $|a_n|^{\frac{1}{n}} < R < 1$.

Application Note: The root test is mostly useful where something (not necessarily a constant) is raised to the n^{th} power.

I **STRONGLY** recommend that you don't try to use this. If you suspect that the limit of the ratio is diverging it is easier to use the n^{th} term test - the n^{th} term should diverge. We will illustrate this with some examples.

Typically it will be easier to apply the ratio test than the root test. I'll give a couple examples of the root test.

Example 2a:

$$\sum \frac{1}{2^k}$$

Example 2b:

$$\sum n^p r^n$$

Example 2c:

$$\sum \frac{(2n)!x^n}{n!n!}$$

Example 2d:

$$\sum \frac{1}{k^p}$$

Example 2e: CAUTION!

$$\sum a_k$$

where the terms a_k are given by

$$a_k = \begin{cases} \frac{1}{k(k-1)} & k \text{ odd} \\ \frac{1}{k!} & k \text{ even} \end{cases}$$

Example 2f:

$$\sum \left(\frac{1 + \ln(n)}{1 + 2\ln(n)} \right)^n$$

Example 2f:

$$\sum \left(\frac{2 + (-1)^n}{4} \right)^n$$