Power Series: 8.6

Power series are one of the main reasons for studying series. Many of the elementary functions from precalc and calculus (sines and cosines, exponentials, logarithms, etc.) can be represented as a power series. A power series is (roughly speaking) a polynomial of infinite degree. More precisely a power series is a series of the following form

$$\sum_{k=0}^{\infty} b_k (x-c)^k.$$

The constants b_k are referred to as the coefficients of the power series, and we frequently say that the series above is centered at c.

Example 1: The following are all power series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{2k}}{k!}$$
$$\sum_{k=0}^{\infty} (1+x)^{k^2}$$
$$\sum_{k=0}^{\infty} k! x^k$$

As the are all sums of (integer) powers of $x-c$ multiplied by some coefficient. So things that are **NOT** power series are

$$\sum_{k=0}^{\infty} \frac{(x-k)^k}{k!}$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k (x)^k \sin(kx)}{k!}$$
$$\sum_{k=0}^{\infty} \frac{\sin(kx)}{k}$$

The last is something called a Fourier series which we may discuss a little in this class. It is an extremely important concept for many areas of engineering (especially signal processing) and physics.

The most important question to answer is this: when does a power series converge? Since the series depend on x the answer is, in general, going to depend on x also. You might think that the set of x for which the series converges might be very complicated, but actually for power series this set is very simple. Lets begin with some examples:

Example 1a:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 1b:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{2k}}{k^3}$$

Example 1c:

$$\sum_{k=0}^{\infty} k! x^k$$

Example 1d:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{2k}}{k}$$

A series may converge for all x, or for no x, more generally a power series will converge in a circle. The radius of this circle is known as the "radius of convergence."

Theorem 6.1: A power series has three possibilities for convergence:

- The series converges absolutely for all real x. (i.e. The radius of convergence is ∞)
- The series converges **ONLY** for $x = c$ (The radius of convergence is zero).
- The series converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$. (The radius of convergence is R).

NOTE: On the radius of convergence the series must be tested. It is possible that the series converges at all points on the boundary, or that it diverge at all points on the boundary, or that it converge at some points an not at others.

The obvious question to ask is whether or not a series can be differentiated. The following answers this question:

Theorem: A power series can be differentiated and integrated termwise within the radius of convergence: that is if

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \qquad |x-c| < R$$

then the derivative is given by

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} |x-c| < R$$

Practice Exercises: Power series
Find the radii of convergence of the following series:

$$\sum_{k=1}^{\infty} k^3(x-2)^k$$
$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k!k!}$$
$$\sum_{k=1,\quad k\ odd}^{\infty} \frac{(x-2)^k}{2^{k^2}}$$
$$\sum_{k=1} \frac{x^k}{k}$$
$$\sum_{k=1} \frac{k^k}{k!} x^k$$
$$\sum_{k=1} \frac{k^k}{(2k)!} x^k$$