

Power Series: 8.6/ Taylor Series 8.7

Recall from last time that a power series may converge for all x , or for no x , more generally a power series will converge in a circle. The radius of this circle is known as the “radius of convergence.”

Theorem 6.1: A power series has three possibilities for convergence:

- The series converges absolutely for all real x . (i.e. The radius of convergence is ∞)
- The series converges **ONLY** for $x = c$ (The radius of convergence is zero).
- The series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$. (The radius of convergence is R).

NOTE: On the radius of convergence the series must be tested. It is possible that the series converges at all points on the boundary, or that it diverge at all points on the boundary, or that it converge at some points an not at others.

The obvious question to ask is whether or not a series can be differentiated. The following answers this question:

Theorem: A power series can be differentiated and integrated termwise within the radius of convergence: that is if

$$f(x) = \sum_{k=0}^{\infty} a_k(x - c)^k \quad |x - c| < R$$

then the derivative is given by

$$f'(x) = \sum_{k=1}^{\infty} k a_k(x - c)^{k-1} \quad |x - c| < R$$

Some examples of this is as follows:

Example: Sum the power series

$$\sum_{k=1}^{\infty} kx^{k-1}$$

What is the radius of convergence?

Example: Sum the power series

$$\sum_{k=1}^{\infty} \frac{x^{k+1}}{k+1}$$

What is the radius of convergence?

Example: Sum the power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

What is the radius of convergence?

Taylor Series

The above special series are all examples of Taylor series. The idea behind Taylor series is that there is a power series attached to all “nice” functions.

To motivate, let’s consider a polynomial function:

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

First note that

$$a_0 = f(0)$$

and that

$$a_1 = f'(0)$$

similarly

$$2a_2 = f''(0)$$

so for a polynomial function $f(x)$ we have

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f}{dx^k}(0) x^k$$

This is true of a much larger class of functions, and motivates the following definition:

Definition: Given a function $f(x)$ the Taylor series for f about the point c is defined as

$$\sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} \frac{d^k f}{dx^k}(c)$$

Example: Suppose that $f(x) = e^x$ and $c = 0$. Note that $f'(x) = e^x$, $f''(x) = e^x$ and thus the exponential function has the following Taylor series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

It is easy to check using the ratio test that this series converges for all values of x .