Taylor Series 8.7

Definition: Recall that, given an infinitely differentiable function $f(x)$ the Taylor series for $f(x)$ about the point c is

$$\sum_{k=0}^{\infty} \frac{d^k f}{dx^k}(c)\frac{(x-c)^k}{k!}$$

Example: Suppose $f(x) = \ln(x)$. Then

$$\begin{aligned}
f'(x) &= \frac{1}{x} \\
f''(x) &= -\frac{1}{x^2} \\
f'''(x) &= \frac{2}{x^3} \\
f''''(x) &= \frac{-3\times 2}{x^4} \\
f^(v)(x) &= \frac{4\times 3\times 2}{x^5} \\
f^k(x) &= (-1)^{k+1}\frac{(k-1)!}{x^k}
\end{aligned}$$

The Taylor series for $f(x) = \ln(x)$ about the point $c = 2$ is

$$\sum_{k=0}^{\infty} \frac{d^k f}{dx^k}(c)\frac{(x-c)^k}{k!} = \ln(2) + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(k-1)!}{k!}\frac{(x-2)^k}{2^k}$$

It is easy to check that the radius of convergence of this series is 2.

Example: Suppose $f(x) = \cos(x)$. Then

$$\begin{aligned}
f'(x) &= -\sin(x) \\
f''(x) &= -\cos(x) \\
f'''(x) &= \sin(x) \\
f''''(x) &= \cos(x)
\end{aligned}$$

The Taylor Series about the point $c = 0$ is given by

The Taylor series about the point $c = 1$ is

$$\cos(1) - \sin(1)(x-1) - \cos(1)(x-1)^2/2! + \sin(1)(x-1)^3/3! + \cos(1)(x-1)^4/4! + \ldots$$

The main idea behind Taylor series is that for "reasonable" functions the Taylor series for the function converges to the function itself:

Big Idea: For "reasonable" functions $f(x)$ the Taylor series converges to $f(x)$:

$$f(x) = \sum_{k=0}^{\infty} \frac{d^k f}{dx^k}(c) \frac{(x-c)^k}{k!} \qquad |x-c| < R$$

Of course, the obvious question is "what is a reasonable function?" . All of the common functions you consider in calculus (sines, cosines, exponentials, logarithms, etc) have Taylor series which converge to the function itself in some radius of convergence. For the sine, cosine, and exponential the radius of convergence is infinity: they converge for all values of x.

Theorem: Suppose that $f(x)$ has $N+1$ derivatives for $x \in (c-r, c+r)$. Let $P_N(x)$ be the first N terms of the Taylor series

$$P_N(x) = \sum_{k=0}^{N} \frac{d^k f}{dx^k}(c) \frac{(x-c)^k}{k!}$$

and $R_N(x)$ be the error $P_N(x) - f(x)$. Then we have the equality

$$R_N(x) = f^{N+1}(z) \frac{(x-c)^{N+1}}{(N+1)!}$$

for some $z \in (x, c)$

Theorem: The Taylor series for the exponential converges to the exponential function on $(0,1)$. **Proof:** Note that $f^{N+1}(z)$ is bounded. It is not hard to see that $\frac{(x-c)^{N+1}}{(N+1)!}$ converges to zero as $N \to 0$. Look at the sum

$$\sum \frac{(x-c)^k}{k!}$$

This converges for all x by the ratio test. By the k^{th} term test we know that the k^{th} term must go to zero. Thus $\frac{(x-c)^N}{N!} \to 0$ as $N \to \infty$. By the above theorem $f(x) - P_N(x) = f^{N+1}(z)\frac{(x-c)^{N+1}}{(N+1)!}$. Letting $N \to \infty$ shows $f(x) =$

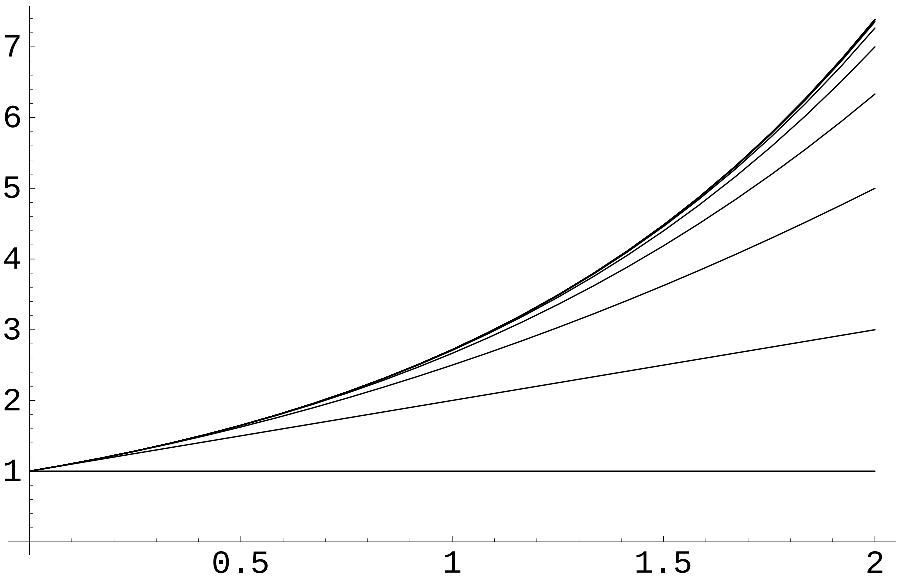

Figure 1: The Taylor polynomials for the exponential $P_0(x) \ldots P_{10}(x) and e^x$

$\lim_{N\to\infty} P_N(x)$.