Taylor Series 8.7 and Applications 8.8

Definition: Recall that, given an infinitely differentiable function $f(x)$ the Taylor series for $f(x)$ about the point c is

$$\sum_{k=0}^{\infty} \frac{d^k f}{dx^k}(c)\frac{(x-c)^k}{k!}$$

Big Idea: For "reasonable" functions $f(x)$ the Taylor series converges to $f(x)$:

$$f(x) = \sum_{k=0}^{\infty} \frac{d^k f}{dx^k}(c)\frac{(x-c)^k}{k!} \qquad |x-c| < R$$

Of course, the obvious question is "what is a reasonable function?" . All of the common functions you consider in calculus (sines, cosines, exponentials, logarithms, etc) have Taylor series which converge to the function itself in some radius of convergence. For the sine, cosine, and exponential the radius of convergence is infinity: they converge for all values of x.

Theorem: Suppose that $f(x)$ has $N+1$ derivatives for $x \in (c-r, c+r)$. Let $P_N(x)$ be the first N terms of the Taylor series

$$P_N(x) = \sum_{k=0}^{N} \frac{d^k f}{dx^k}(c)\frac{(x-c)^k}{k!}$$

and $R_N(x)$ be the error $P_N(x) - f(x)$. Then we have the equality

$$R_N(x) = f^{N+1}(z)\frac{(x-c)^{N+1}}{(N+1)!}$$

for some $z \in (x, c)$

Theorem: The Taylor series for the exponential converges to the exponential function on $(0,1)$. **Proof:** Note that $f^{N+1}(z)$ is bounded. It is not hard to see that $\frac{(x-c)^{N+1}}{(N+1)!}$ converges to zero as $N \to 0$. Look at the sum

$$\sum \frac{(x-c)^k}{k!}$$

This converges for all x by the ratio test. By the k^{th} term test we know that the k^{th} term must go to zero. Thus $\frac{(x-c)^N}{N!} \to 0$ as $N \to \infty$. By the above theorem $f(x) - P_N(x) = f^{N+1}(z)\frac{(x-c)^{N+1}}{(N+1)!}$. Letting $N \to \infty$ shows $f(x) = \lim_{N\to\infty} P_N(x)$.

One thing to realize is that once we have a few Taylor series under our belt we can use these to derive new ones:

Example: Find the Taylor series for $f(x) = \frac{(e^x - 1)}{x}$

This is difficult if we use the formula. But if we use the fact that we already know the Taylor series for the exponential:

Example: Find the Taylor series for $f(x) = \frac{\sin(x^2) - x^2}{x^6}$

Example: Find the Taylor series for

$$f(x) = \arctan(x)$$

Here is a possibly useful table of some common functions together with their Taylor series...

Function $f(x)$	Point expanded about	Taylor Series	Radius of Convergence
e^x	0	$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	∞
$\sin(x)$	0	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$	∞
$\cos(x)$	0	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	∞
$\frac{1}{1+x}$	0	$\sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \dots$	1
$\frac{1}{1+x}$	c	$\sum_{k=0}^{\infty} (-1)^k \frac{(x-c)^k}{c^k} = 1 - \frac{(x-c)}{c} + \frac{(x-c)^2}{c^2} - \dots$	c
$\ln(x)$	0	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$	1
$\arctan(x)$	0	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$	1

Taylor series are useful for the following reason: they show us that any **reasonable** function can be approximated by a polynomial. We have error estimates, so we can decide how many terms we need to take to get a good approximation. The nice thing is that polynomials are easy to work with (integrate, differentiate, take limits). This is the basic point behind section 8.8.

Example: Find the $cos(.75)$ using a Taylor series to within 10^{-7}.

There are a couple of ways of doing this. The simplest is to use the Taylor series with $c = 0$:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

A better way is to use the Taylor series about the point $\pi/4 \approx .785$

Example: Compute the following limit:

$$\lim_{x\to 0} \frac{\sin(x^2) - x^2}{x^6}$$

Note: It is not hard, though somewhat tedious, to calculate this using (for example) L'Hopital's rule. It is much easier to use the Taylor series to compute the limit.

Example: Compute the limit

$$\lim_{x\to 0} \frac{\arctan(x) - x}{x}$$

Example: Compute the following integral approximately (to within 10^{-5}) using Taylor series:

$$\int_0^1 \frac{e^x - e^{-x}}{x}$$