Taylor Series Applications 8.8 and Fourier Series 8.9

Definition: A function is periodic with period T if $f(x+T) = f(x)$.

Example: The functions $\sin(x), \cos(x)$ are periodic with period 2π. The functions $\sin(Mx), \cos(Mx)$ are periodic with period $2\pi/M$

Definition: A Fourier series is a series of the following form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

Note that this series converges if **both** series

$$\sum_{k=1}^{\infty} a_k$$
$$\sum_{k=1}^{\infty} b_k$$

converge.

Note that $f(x)$ is necessarily 2π periodic since

$$\begin{aligned} f(x+2\pi) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k(x+2\pi)) + b_k \sin(k(x+2\pi)) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \\ &= f(x) \end{aligned}$$

The big idea of Fourier series is that (roughly speaking) every periodic function can be written as a Fourier series. It is rather technical to state this precisely, so I won't.

This idea is common in music. When a musical instrument produces a particular note it is making a sound which has a particular period. for instance the A above middle C has frequency 440 Hz, meaning that it repeats 440 times a second (the period is 1/440 seconds). MOst instruments do not produce a pure tone. Rather they produce a tone which is periodic but not a pure sine wave. The higher tones are known as overtones.

An example waveform for the Euphonium is given at
http://hyperphysics.phy-astr.gsu.edu/hbase/music/euph.html#c1
and a similar one for the saxophone:
http://hyperphysics.phy-astr.gsu.edu/hbase/music/saxw.html#c1
and the flute
http://hyperphysics.phy-astr.gsu.edu/hbase/music/flutew.html#c1

Fourier Analysis There are basically two sides to Fourier series: Fourier synthesis and Fourier Analysis. Fourier synthesis is the question: given a_k how do I find $f(x)$. I've already answered that question. The converse is this: given

$f(x)$ how do I find a_k. For instance given the waveform for the saxophone or the Euphonium how do I compute the coefficients a_k. The answer is simple:

Theorem: Given

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

then the a_k are related to $f(x)$ by

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx \end{aligned}$$

PROOF: Orthogonality

Examples: Square Wave The square wave is defined by