Parametric Plotting

In precalculus we learn to plot the graph of a function, but there are curves which are not the graph of any function which are, nevertheless, interesting. A more general notion than the graph of a function is that of a parametric curve.

A parametrc curve is something of the form

$$\begin{aligned} x(t) &= f_1(t) \\ y(t) &= f_2(t) \end{aligned}$$

As in the case of the graph of a function for each value of t (the parameter, whence parametric) there is a unique x and y value. Note that a parametric curve need not pass the **VERTICAL LINE TEST**. There can, in general, be more than one value of t which gives the same value of x.

Example: The parametric curve

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \end{aligned}$$

represents a circle of radius 1. One way to see this is to note that

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

Another is to note that this more or less follows from the definition of the sine and the cosine as the length of the opposite and adjacent sides respectively (assuming that the hypotenuse is 1). Note that the circle **FAILS** the vertical line test: it is not the graph of a function (although it is the graph of two functions: $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$.

Note that the parametric curve

$$\begin{aligned} x(t) &= \sin(t) \\ y(t) &= \cos(t) \end{aligned}$$

Represents the same curve $x^2 + y^2 = 1$ but they are traced out in opposite directions: the first is traced out clockwise and the second is traced out counterclockwise.[1]

It is interesting to do a few simple operations on this curve. First we can add 1 to the x coordinate

$$\begin{aligned} x(t) &= 1 + \sin(t) \\ y(t) &= \cos(t) \end{aligned}$$

You can notice that $(x-1)^2 + y^2 = \sin^2(t) + \cos^2(t) = 1$ but it is easier to think this way: we are shifting all of the x coordinates by a fixed amount 1 so

[1] Mathematicians always like to trace out curves counterclockwise

this is the same as the original curve but shifted over by 1

Similarly we can add 3 to the y coordinate

$$\begin{aligned} x(t) &= \sin(t) \\ y(t) &= 3 + \cos(t) \end{aligned}$$

and this is the same curve but shifted by 3 in the y direction.

We can also think about other transformations. For instance if we multiply the x coordinate by 2 it simply amounts to a stretch of the x coordinate: the new curve is an ellipse.

In the case above we can find an equation for the curve $x^2+y^2=1$ but more often than not we will not be able to do that. A few examples:

Example: The parametric curve

$$\begin{aligned} x(t) &= t^3 - t + 1 \\ y(t) &= t^3 - 18t^2 + 9 \end{aligned}$$

Is going to difficult to write in the form $f(x,y) = 0$, but we can graph it just fine. The graph of the curve looks like

Example: For the parametric curve

$$\begin{aligned} x(t) &= t^4 - 2t^2 + 11 \\ y(t) &= t^2 - 5 \end{aligned}$$

we can eliminate the variable t to find

$$x(t) = (y+5)^2 - 2(y+5) + 11 = y^2 + 8y + 26$$

This represents a parabola turned on it's side. Note that there is a little bit of subtlety here though. If I plot these two curves I **don't** quite get the same thing. The parametric curve only gives me a piece of the (nonparametric) curve. The reason for this is as follows: Looking at the parametric form it is clear that $y \geq -5$. But when I plotted the sideways parabola I didn't put **any** restriction on y. So I got an extra "piece" of the solution which correspond to $y < -5$ (t *imaginary*)

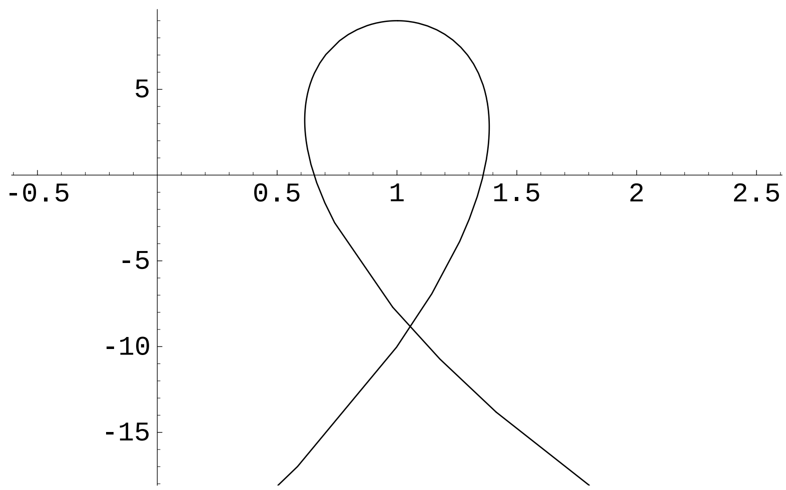

Figure 1: A plot of the parametric curve $x(t) = t^3 - t + 1, y(t) = t^3 - 18t^2 + 9$.

Example: Lissajous figures

If any of you have played around with an oscillascope or watched corny old sci-fi movies you've probably seen a Lissajous figure. The Lissajous figure is the curve

$$\begin{aligned} x(t) &= \cos(t + \phi) \\ y(t) &= \cos(\alpha t) \end{aligned}$$

where α is a rational number. An example is $\alpha = 5/3$ which gives the following plot

It is not hard to show that this curve closes whenever α is a rational number. It is interesting to plot this curve when α is an irrational number. Here is a plot of this curve when $\alpha = \sqrt{2}$ for t between $[0, 600]$.

It can be shown that if α is irrational then this curve **never** closes. In fact the curve passes arbitrarily close to every point in the square.

Calculus and Parametric Curves (9.2)

Since this is a calculus class we need to be able to do calculus on parametric curves. The first thing that we need to define is the tangent vector.

Def: Given a parametric curve $\gamma(t) = (x(t), y(t))$ the tangent vector is given by $\gamma'(t) = (\frac{dx(t)}{dt}, \frac{dy(t)}{dt})$

The tangent vector represents the local direction of motion of the curve, as

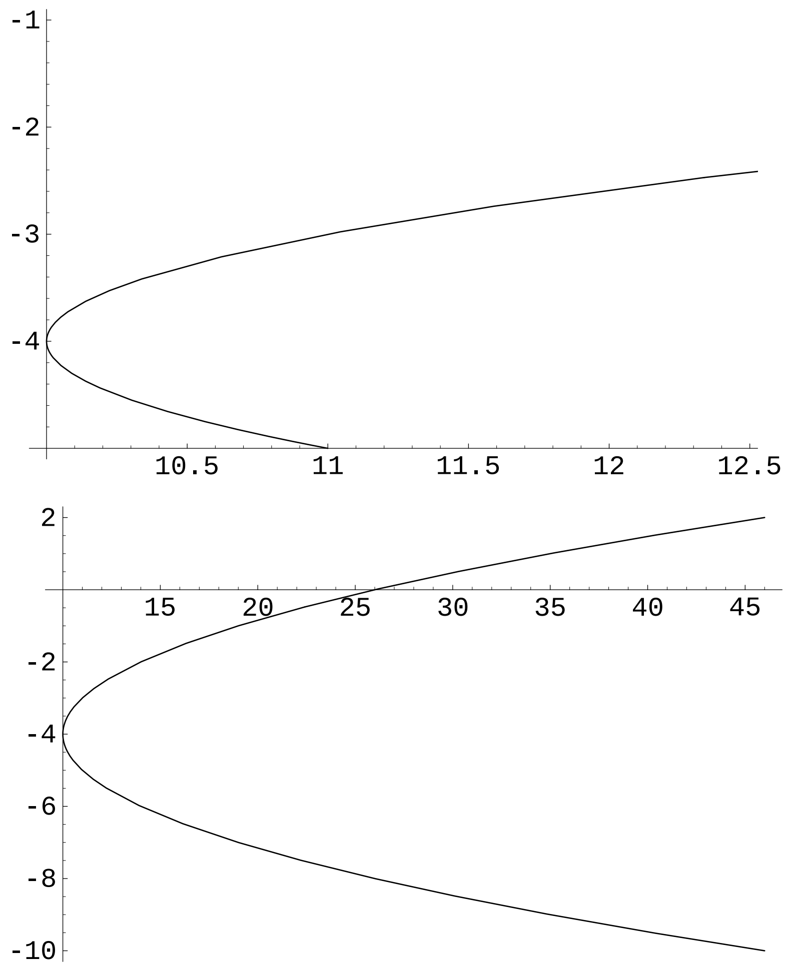

Figure 2: The curves $x(t) = t^4 - 2t^2 + 11 \quad y(t) = t^2 - 5$ and $x(t) = y^2 + 8y + 26$.

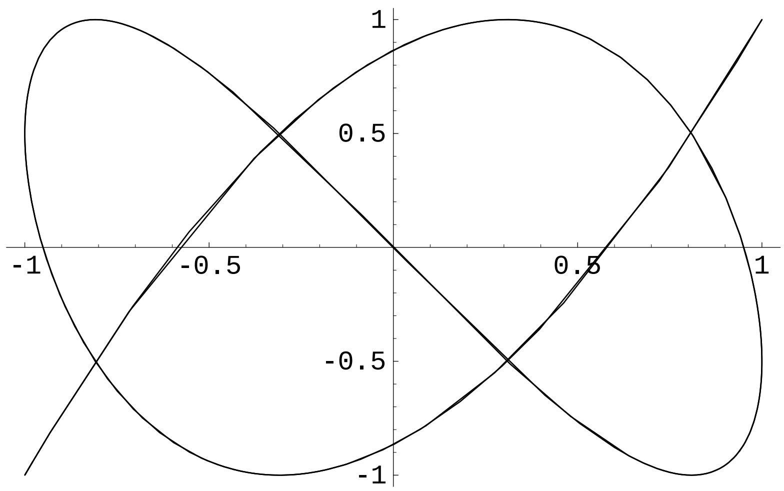

Figure 3: Lissajous figure with $\alpha = \frac{5}{3}$

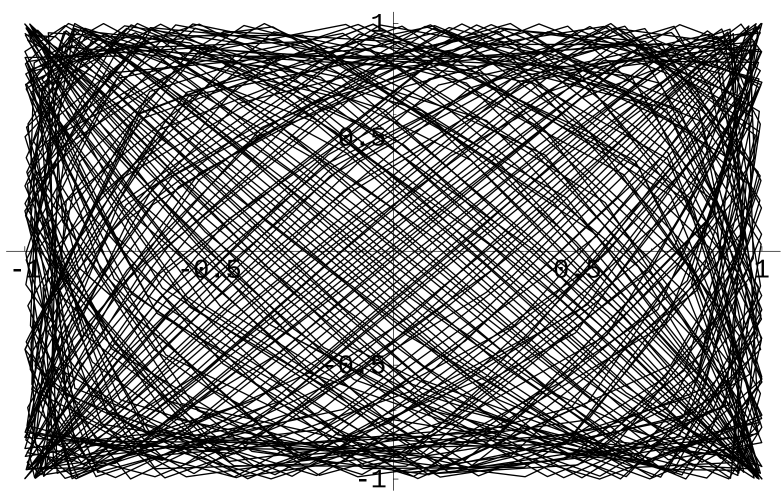

Figure 4: Lissajous figure with $\alpha = \sqrt{2}$

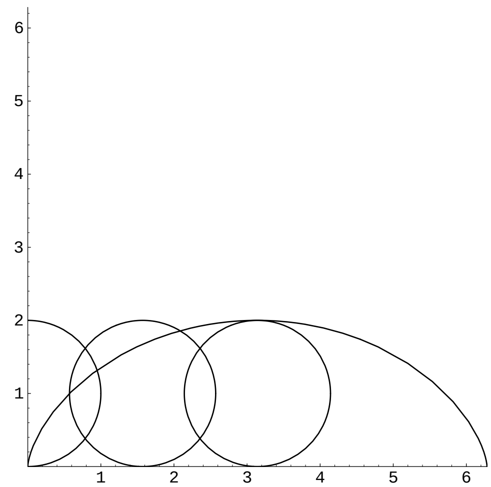

Figure 5: The cycloid curve

can be seen by the following sketch

Example: Cycloid.

The cycloid is the curve which is traced out by a point on the edge of a rolling wheel. It is not hard to find the equation for the cycloid.

$$y(t) = 1 - \cos(t) \qquad x(t) = t - \sin(t)$$