9.2 Parametric Curves II - Calculus

Last time we began discussing how to do calculus on a curve defined parametrically. The first thing we discussed was the tangent vector, which is the natural extension of the derivative of a function. Recall that

Definition: Given a parametric curve defined by $(x(t), y(t))$ the tangent vector (velocity vector) is given by $(\frac{dx}{dt}, \frac{dy}{dt})$.

Again one example we gave last time was the cycloid curve

$$x(t) = t - \sin(t) \qquad y(t) = 1 - \cos(t)$$

The tangent vector is given by

$$(\frac{dx}{dt}, \frac{dy}{dt}) = (1 - \cos(t), \sin(t))$$

Again we note that when $t = 0, 2\pi, 4\pi$ the velocity vector is $(0, 0)$, so the curve is instantaneously at rest. This makes sense - these are the points where the curve turns around. At $t = \pi, 3\pi, 5\pi$ the velocity vector is given by $(\frac{dx}{dt}, \frac{dy}{dt}) = (1 - \cos(t), 0)$. Again this makes sense physically - the motion is all in the x direction with no component in the y direction. This makes sense since the curve has a maximum in the vertical direction at these points.

It is easy to relate this to the *usual* derivative $\frac{dy}{dx}$. By the chain rule we have that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$

which we can solve for $\frac{dy}{dx}$ to give

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

So in terms of the ordinary derivative we have $\frac{dy}{dx} = 0$ at $t = \pi, 3\pi, 5\pi, \ldots$ (which corresponds to $x = \pi, 3\pi, 5\pi, \ldots$) Again this makes sense because the function has a maximum there. The derivative goes to infinity as $x \to 0, 2\pi, 4\pi, \ldots$. This makes sense because (as the picture makes clear) the slope is infinite there.

Example:

Another example is given in the text: The carnival ride called the Scrambler (see Ex 2.1 Section 9.2) has the equation

$$x(t) = 2\cos(t) + \sin(2t) \qquad y(t) = 2\sin(t) + \cos(2t)$$

The tangent vector to the curve has the form

$$\frac{dx}{dt} = 2\cos(2t) - 2\sin(t) \qquad \frac{dy}{dt} = 2\cos(t) - 2\sin(2t)$$

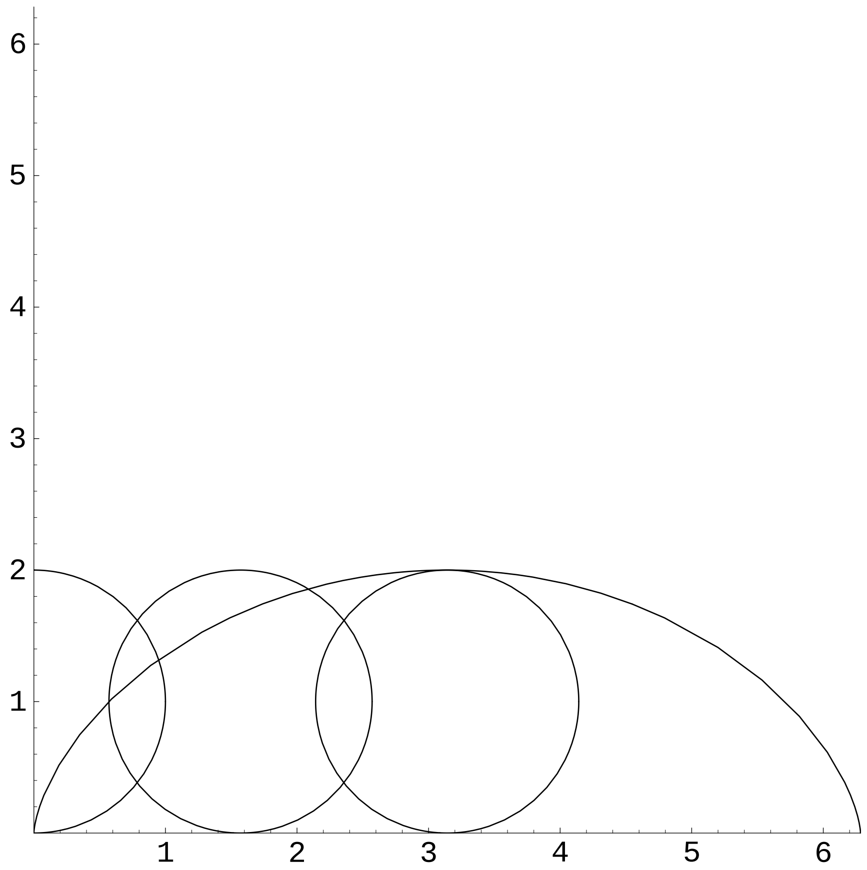

Figure 1: The cycloid curve with derivatives and tangent vectors.

Let's compute the speed at which a point on the curve is moving - the length of the velocity vector. It is easy to see that

$$\begin{aligned}
(\frac{dx}{dt})^2+(\frac{dy}{dt})^2 &= (2\cos(2t)-2\sin(t))^2+(2\cos(t)-2\sin(2t))^2\\
&= 4\cos^2(2t)-8\cos(2t)\sin(t)+4\sin^2(t)+4\cos^2(t)-8\cos(t)\sin(2t)+4\sin^2(2t)\\
&= 8-8\cos(2t)\sin(t)-8\cos(t)\sin(2t)\\
&= 8-8\sin(3t)
\end{aligned}$$

where the last step follows from the identity $\sin(a+b)=\sin(a)\cos(b)+\sin(b)\cos(a)$.

Thus we can see that the speed is zero when $\sin(3t)=1$ which occurs when $3t=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2},\ldots$, or $t=\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\ldots$. The speed is largest $\sqrt{16}=4$ when $\sin(3t)=-1$ which occurs when $t=\frac{\pi}{2},\frac{7\pi}{6},\frac{11\pi}{6},\ldots$.

Area:

It is also useful to be able to find the area enclosed by a (closed) parametric curve. Suppose the parametric equation

$$(x(t),y(t)) \qquad t\in[a,b)$$

defines a closed curve. The the area enclosed is given by

$$Area=\int_a^b y(t)\frac{dx}{dt}dt=-\int_a^b x(t)\frac{dy}{dt}dt$$

There are a couple of reasonably straightforward ways to see this. The first is to relate this to the usual integral:

The second is to draw the following picture

And note that

$$\theta = \arctan(\frac{y}{x})$$

and thus

$$\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x^{-1} - y\frac{dx}{dt}x^{-2}}{x^2 + y^2}$$