Lecture 3:: Trigonometric Substitutions Revisited

0.1 Reduction Formulae:

One idea which is useful is that of a reduction formula. The basic idea is to reduce an integral to a similar integral of lower order. For example: to compute $\int \sin^n(x)$ we might do the following

$$\begin{aligned}
\int \sin^n(x)dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-1}(x)dx \\
n\int \sin^n(x)dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int sin^{n-1}(x)dx \\
\int \sin^n(x)dx &= \frac{-1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx
\end{aligned}$$

Thus we can relate $\int \sin^2(x)dx$ to $\int \sin^0(x)dx = \int dx$, $\int \sin^4(x)dx$ to $\int sin^2(x)dx$. Similarly for odd powers we find that we can relate $\int \sin^3(x)dx$ to $\int \sin(x)dx$, $\int \sin^5(x)dx$ to $\int \sin^3(x)$

Self Study Problems:

- Compute $\int \sin^6(x)dx$ using the above reduction formula.
- Derive an analogous reduction formula for $\int \cos^n(x)dx$.

0.2 General Trig Substitutions:

To start with I will calculate a couple of integrals we'll use a bunch of times in this section. I want to start by noting that

$$\frac{d}{dx}\ln(1+x) - \ln(1-x) = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$$

and thus that

$$\int \frac{1}{1-x^2}dx = \frac{1}{2}(\ln(1+x) - \ln(1-x))$$

We'll see in the next chapter how to do the above integral in a more straightforward way using the method of partial fractions.

I want to begin by evaluating a couple of integrals. The first is done in the book and is

$$\int \frac{1}{\cos(x)}dx = \ln|\sec(x) + \tan(x)|$$

The calculation in the book is a little bit opaque so I am going to use a different method. I'm actually going to do the related integral

$$\int \frac{1}{\sin(x)}dx = \ln|\csc(x) - \cot(x)|$$

It can be done the following way: Write the integral as

$$\int \frac{\sin(x)}{\sin^2(x)}dx = \int \frac{\sin(x)}{1-\cos^2(x)}dx$$

Making the substitution $u = \cos(x)$ gives

$$-\int \frac{1}{1-u^2}du$$

We do this integral using the above and get the result

$$-\int \frac{1}{1-u^2}du = -\frac{1}{2}\ln(\frac{1+u}{1-u}) = -\frac{1}{2}\ln(\frac{1+\cos(x)}{1-\cos(x)})$$

It is not hard to show using some trigonometric identities that

$$-\frac{1}{2}\ln(\frac{1+\cos(x)}{1-\cos(x)}) = \ln|\csc(x) - \cot(x)| = \ln|\tan(x/2)|$$

Self Study Problem: Show that the above trigonometric identities hold.

Consider another example:

$$\int \frac{1}{1+x^2}dx$$

I am going to do this via a trigonometric substitution. I am going to let $x = \tan(u)$. Differentiating gives $dx = \sec^2(u)du$. The integral becomes

$$\int \frac{1}{1+x^2}dx = \int \frac{\sec^2(u)}{1+\tan^2(u)}dx = \int du = u = \arctan(x)$$

Similarly if I wanted to do the integral

$$\int \frac{x^2}{(1+x^2)^2}dx$$

and I make the substitution $x = \tan(u), dx = \sec^2(u)du$ we get an integral of the form

$$\int \frac{\tan^2(u)\sec^2(u)}{\sec^4(u)}du = \int \frac{\tan^2(x)}{\sec^2(x)}dx = \int \sin^2(u)du = \frac{u}{2} - \frac{1}{4}\sin(2u) = \frac{1}{2}\arctan(x) - \frac{1}{4}(whatever)$$

Another example: consider the integral

$$\int \frac{\sqrt{1-x^2}}{x}dx$$

The substitution $x = \cos(u)$ reduces the integral to

$$-\int \frac{\sin^2(u)}{\cos(u)} du = -\int \cos(u) \frac{1}{\cos(u)} du$$

The integral $\int \frac{1}{\cos(u)} du$ was done in recitation (or in example 3.8 on p. 525 of the textbook) and is given by

$$\int \frac{1}{\cos(u)} du = \ln|\sec(x) + \tan(x)|$$

thus we have

$$-\int \frac{\sin^2(u)}{\cos(u)} du = \sin(u) - \ln|\sec(x) + \tan(x)|$$

solving for u in terms of x gives

$$\int \frac{\sqrt{1-x^2}}{x} dx = \sqrt{1-x^2} - \ln|\frac{1+\sqrt{1-x^2}}{x}|$$

As a general rule: play with the integrals and see if you can find a substitution that simplifies them. IN many cases the following are likely to work

- $\sqrt{a^2 - x^2}$ make the substitution $x = a\cos u$ (or $x = \sin(u)$).
- $\sqrt{a^2 + x^2}$ make the subtitution $x = a\tan(u)$
- $\sqrt{x^2 - a^2}$ make the substitution $x = a\sec(u)$

Some more examples:

$$\int \frac{1+x}{x^2\sqrt{1+x^2}} dx$$