

## Lecture 3:: Trigonometric Substitutions Revisited

### 0.1 Rational Functions:

A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P(x), Q(x)$  are polynomials (and thus  $f(x)$  is the Ratio of two polynomials). The integrals of such functions can always be done *assuming* that the denominator  $Q(x)$  can be factored.

Let me remind you of some basic facts about polynomials: A polynomial  $Q(x) = A_n x^n + A_{n-1} x^{n-1} + \dots A_1 x + A_0$  admits a factorization of the form  $Q(x) = A_n (x - x_1)(x - x_2) \dots (x - x_n)$ , where  $x_i$  are the roots (or zeros) of  $Q(x)$ . The  $x_i$  may not be distinct. If they are the polynomial is said to have distinct roots. Otherwise a root which appears several times is said to be a multiple root. The  $x_i$  are not necessarily real numbers. If *all* of the roots are real the polynomial is said to have real roots. The degree of the polynomial is the highest (non-zero) power of  $x$  which appears.

### 0.2 Example:

The polynomials below admit the following factorizations

$$Q(x) = x^3 - 3x^2 + 2x = x(x - 1)(x - 2)$$

$$Q(x) = x^3 - x^2 - x + 1 = (x - 1)(x - 1)(x + 1)$$

$$Q(x) = x^3 - 2x^2 + 3x - 2 = (x - 1)\left(x - \frac{1 + \sqrt{7}i}{2}\right)\left(x - \frac{1 - \sqrt{7}i}{2}\right)$$

The first polynomial has distinct real roots. The second polynomial has real root that are not distinct, with  $-1$  as a simple root and  $1$  as a double root. The third polynomial has distinct roots which are complex. All polynomials are third degree, and thus have three roots (assuming that we count correctly - in the second example since  $1$  appears twice we count it twice when we are counting roots).

We begin with the simplest case, which is when the roots are real and distinct and the degree of the numerator is less than the denominator.

### 0.3 Example

Calculate

$$\int \frac{x - 5}{x^2 - 3x + 2} dx$$

First note that the denominator can be factored as  $x^2 - 3x + 2 = (x - 1)(x - 2)$ . Thus we have  $\frac{x-5}{x^2-3x+2} = \frac{x-5}{(x-1)(x-2)}$ . We look for a representation of the form

$$\frac{x-5}{(x-1)(x-2)} = \frac{a}{(x-1)} + \frac{b}{(x-1)}$$

for some *constants*  $a, b$ . Rationalizing the denominators gives

$$a + b = 1 \quad 2a + b = 5$$

Solving these equations simultaneously gives  $a = 4, b = -3$ . Thus we have the representation

$$\frac{x-5}{(x-1)(x-2)} = \frac{4}{(x-1)} - \frac{3}{(x-1)}$$

and

$$\int \frac{x-5}{x^2-3x+2} dx = 4 \ln|x-1| - 3 \ln|x-2| = \ln \left| \frac{(x-1)^4}{(x-2)^3} \right|$$

#### 0.4 Example

Integrate

$$\int \frac{x-1}{x^3-5x^2+6x}$$

First note that one can factor off a common factor of  $x$ , yielding  $x(x^2-5x+6)$ . The quadratic formula gives the roots of the latter as 2, 3, and thus we have  $x(x-2)(x-3)$ . Setting up the problem

$$\frac{x-1}{(x)(x-2)(x-3)} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x-3}$$

rationalizing the denominators gives

$$\begin{aligned} a(x-2)(x-3) + bx(x-3) + cx(x-2) &= x-1 \\ (a+b+c)x^2 + (-5a-3b-2c)x + 6a &= x-1 \end{aligned}$$

Thus

$$\begin{aligned} a + b + c &= 0 \\ -5a - 3b - 2c &= 1 \\ 6a &= -1 \end{aligned}$$

Thus  $a = -\frac{1}{6}, b = -\frac{1}{2}, c = \frac{2}{3}$

**In General:**

In general it should be clear that

$$\frac{a_1}{x-x_1} + \frac{a_2}{x-x_2} + \dots + \frac{a_n}{x-x_n}$$

is going to give a rational function of the form

$$\frac{P(x)}{(x-x_1)(x-x_2)\dots(x-x_n)}$$

where  $P(x)$  is of degree  $(n-1)$  or lower. So the general scheme I outlined will work if the degree of  $P(x)$  is less than the degree of  $Q(x)$ . What happens when the degree of  $P(x)$  is greater than the degree of  $Q(x)$ ? In this case it is always easy to reduce the problem to one where the degree of the numerator is less than the degree of the denominator.

### 0.5 Example:

Integrate

$$\int \frac{x^4 - 5x^2}{x^3 - 3x^2 + 2x} dx$$

To do this integral we first note that the denominator is of degree 3 and the numerator degree 4, so the procedure outlined above is not going to work without some modification. Fortunately the modification is small. I claim that we can write the integrand in the following form

$$\frac{x^4 - 5x^2}{x^3 - 3x^2 + 2x} = ax + b + \frac{R(x)}{x(x-1)(x-2)}$$

The procedure for doing this is known as synthetic division. (Show) Thus we get

$$\frac{x^4 - 5x^2}{x^3 - 3x^2 + 2x} = x + 3 + \frac{2x^2 - 6x}{x(x-1)(x-2)}$$

**Exercise:** Show that

$$\int \frac{x^4 - 5x^2}{x^3 - 3x^2 + 2x} dx = \frac{x^2}{2} + 3x + 4 \ln|x-1| - 2 \ln|x-2|$$

## 1 Repeated Roots:

In the case a root is of higher multiplicity, the procedure changes a little but it still basically the same. Suppose that  $x_1$  is a root of multiplicity 3. For example

$$\int \frac{x^2 - x + 7}{x^4 - x^3} dx = \int \frac{x^2 - x + 7}{x^3(x-1)} dx$$

you would look for an expansion of the form

$$\begin{aligned}\int \frac{x^2 - x + 7}{x^3(x-1)} dx &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{d}{x-1} \\ &= \frac{ax^2 + bx + c}{x^3} + \frac{d}{x-1}\end{aligned}$$

The two equations are equivalent. The book presents it in the first way. I learned and prefer the second way. You may do it any way you choose.

### 1.1 Example:

Do the above problem:

Starting with

$$\int \frac{x^2 - x + 7}{x^3(x-1)} dx = \frac{ax^2 + bx + c}{x^3} + \frac{d}{x-1}$$

we get

$$\begin{aligned}\int \frac{x^2 - x + 7}{x^3(x-1)} dx &= \frac{(ax^2 + bx + c)(x-1)}{x^3(x-1)} + \frac{dx^3}{x^3(x-1)} \\ &= \frac{ax^3 + (b-a)x^2 + (c-b)x - c}{x^3(x-1)} + \frac{dx^3}{x^3(x-1)}\end{aligned}$$

giving  $a+d=0$ ,  $b-a=1$ ,  $c-b=-1$ ,  $-c=7$  or  $c=-7$ ,  $b=-6$ ,  $a=-7$ ,  $d=7$ . Thus

$$\int \frac{x^2 - x + 7}{x^3(x-1)} dx = \int \frac{-7}{x} - \frac{6}{x^2} - \frac{7}{x^3} + \frac{7}{x-1} dx = -7 \ln|x| + \frac{6}{x} + \frac{7}{2}x^{-2} + 7 \ln|x-1|$$