0.1 Partial Fractions: Distinct Real roots (Simplified case)

0.2 Example:

Evaluate

$$\int \frac{2x+1}{x^2+4x+3} dx$$

The partial fractions expansion is

$$\frac{2x+1}{x^2+4x+3} = \frac{a}{x+1} + \frac{b}{x+3}$$

Multiplying through by $x^2+4x+3 = (x+1)(x+3)$ gives

$$2x+1 = a(x+3) + b(x+1)$$

Setting $x = -1$ gives $2a = -1$. Setting $x = -3$ gives $-2b = -5$. Thus we have

$$\frac{2x+1}{x^2+4x+3} = -\frac{1}{2(x+3)} + \frac{5}{2(x+1)}$$

and

$$\int \frac{2x+1}{x^2+4x+3} dx = \int -\frac{1}{2(x+3)} + \frac{5}{2(x+1)} = -\frac{1}{2} \ln|x+3| + \frac{5}{2} \ln|x+1| + C$$

0.3 Partial Fractions: Complex Roots

The final case to consider is the case when the polynomial has complex roots. We are assuming that the polynomial has real coefficients. In this case the complex roots appear in complex conjugate pairs. Lets assume that there are r real roots and $n-r$ complex roots (note that $n-r$ must be even). In this case the polynomial admits a factorization of the form

$$Q(x) = A_n(x-x_1)(x-x_2)\ldots(x-x_r)(x^2+B_1x+C_2)(x^2+B_2x+C_2)\ldots(x^2+B_{\frac{n-r}{2}}x+C_{\frac{n-r}{2}})$$

So we get a bunch of linear factors corresponding to the real roots, and a bunch of *quadratic* factors corresponding to the complex conjugate pairs of roots. In this case the partial fractions decomposition takes the form

$$\frac{P(x)}{Q(x)} = \frac{a_1}{x-x_1} + \frac{a_2}{x-x_1} + \ldots \frac{a_r}{x-x_1} + \frac{b_1x+c_1}{x^2+B_1x+C_1} + \frac{b_2x+c_2}{x^2+B_2x+C_2} \cdots \frac{b_{\frac{n-r}{2}}x+c_{\frac{n-r}{2}}}{x^2+B_{\frac{n-r}{2}}x+C_{\frac{n-r}{2}}}$$

Essentially one has the reciprocal of a linear function for each distinct real root and the ratio of a linear function and a quadratic function.

0.4 Example

Integrate

$$\int \frac{2x+3}{x^3+x}dx$$

Note that the denominator is given by $x^3 + x = x(x^2 + 1)$, so it has one real root $x = 0$ and two imaginary roots $x = \pm i$. We look for a partial fractions expansion in the form

$$\frac{2x+3}{x^3+x} = \frac{a}{x} + \frac{bx+c}{x^2+1}$$

Multiplying through by the denominator gives

$$\underset{=}{\underline{2x+3}} a(x^2+1) + (bx+c)(x)$$

Giving the three equations

$$\begin{aligned} a &= 3 \\ c &= 2 \\ a+b &= 0 \end{aligned}$$

Thus we get

$$\begin{aligned} \frac{2x+3}{x^3+x} &= \frac{3}{x} + \frac{-3x+2}{x^2+1} \\ \int \frac{2x+3}{x^3+x} &= 3\ln|x| - \frac{3}{2}\ln(x^2+1) + 2\arctan(x) + C \end{aligned}$$

Note that one can do the same thing one does in the case of real distinct roots and make the substitutions $x = 0, x = i, x = -i$ to get three equations. In the complex case, however, that the equations for b, c are coupled.

1 Improper Integrals:

Lets start by considering the following integral:

$$\int_{-2}^{2} \frac{1}{x^4}dx = \frac{-1}{3}x^{-3}|_{-2}^{2} = \frac{-1}{3}(8 - (-8)) = -\frac{16}{3}$$

Of course this result is nonsense. One way to see this is that we are integrating a positive quantity $\frac{1}{x^4}$, but the result is negative. This can't be - a sum of positive things must be positive.

Of course the problem is that the fundamental theorem requires that the integrand be continuous over the interval of integration. The integrand here diverges at $x = 0$, and is not continuous. So the fundamental theorem does not hold.

We come across a lot of integrals in practice where the integrand is not well-defined at some point (frequently the endpoint) but one would nonetheless like to attach a value to it. This is where the concept of an improper integral arises.

Definition: If f is continuous on $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$ then we define the improper integral

$$\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx$$

If the limit exists, in which case we say the integral converges. If the limit does not exist we say the integral diverges. Similarly for the lower endpoint.

Examples: Evaluate the convergence of the following integrals:

$$\int_0^{\frac{1}{2}} \frac{dx}{x \ln|x|}$$

$$\int_0^{\frac{1}{2}} \frac{dx}{x \ln^2|x|}$$

First note that $\lim_{x \to 0} x \ln|x| = 0$ and $\lim_{x \to 0} x \ln^2|x| = 0$ by L'Hopital's rule (**Exercise: Check this!**) so the integrand diverges as $x \to 0$. Both integrals can be done by the simple substitution $u = \ln(x)$. In the first case we find that

$$\int_R^{\frac{1}{2}} \frac{dx}{x \ln|x|} = \ln|\ln|x|||_R^{\frac{1}{2}}$$

As $R \to 0$ we find that $\ln|x| \to -\infty$ and thus $\ln|\ln|x|| \to \infty$. Thus the integral diverges.

In the second case we find that

$$\int_0^{\frac{1}{2}} \frac{dx}{x \ln^2|x|} = -\frac{1}{\ln|x|}|_R^{\frac{1}{2}}$$

as $R \to 0$ we find that $\ln|x| \to \infty$ and thus $\frac{1}{\ln|x|} \to 0$. Thus as $R \to 0$ the integral approaches a definite limit given by

$$\int_0^{\frac{1}{2}} \frac{dx}{x \ln^2|x|} = \frac{1}{\ln|2|}.$$