**Lecture 5 Practice Exercises**

Evaluate the following integrals using the method of partial fractions and any additional methods or substitutions that you might need

$$\int \frac{1}{(x^3+5x^2+12x+8)}dx$$
$$\int \frac{1+x}{x(x^4+2x^2+1}dx$$

**Hint:** In the first example the denominator vanishes at $x = -1$.

Find the proper form for the partial fractions expansion of the following rational functions. You do not need to actually calculate the coefficients.

$$\frac{2x^5+1}{(x-1)^3(x^4+2x^2+1)}$$
$$\frac{2}{(x+1)^3(x^4+2x^2+1)^2}$$
$$\frac{5x-7}{x^4+6x^2+5}$$

Compute the limit:

$$\lim_{x\to 0} x \ln^n |x|$$

where $n$ is an integer. If you cannot do the general case try $n = 1, 2$.

For what values of $\alpha$ does the integral

$$\int_0^1 x^{-\alpha} dx$$

converge?

For what values of $\beta$ does the integral

$$\int_0^{\frac{1}{2}} \frac{1}{x \ln^\beta |x|} dx$$

converge?

# LECTURE 5

$x^3+5x^2+12x+8$ vanishes @ $x=-1$

Thus

$$\begin{array}{r|l} & x^2+4x+8 \\ \hline x+1 & x^3+5x^2+12x+8 \\ & \underline{x^3+x^2} \\ & \quad 4x^2+12x \\ & \quad \underline{4x^2+4x} \\ & \qquad 8x+8 \end{array}$$

by completing square $x^2+4x+8=(x+2)^2+4$

$$\frac{1}{x^3+5x^2+12x+8} = \frac{1}{(x+1)((x+2)^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{(x+2)^2+4}$$

$$A\left((x+2)^2+4\right)+(Bx+C)(x+1)=1$$

$5A=1 \qquad A=\frac{1}{5} \qquad (@\,x=-1)$

$8A+C=1 \qquad (@\,x=0)$

$C=1-8A=1-\frac{8}{5}=\frac{-3}{5}$

$4A+(C-2B)(-1)=1 \qquad @(x=-2)$

$2B=1+C-4A = 1-\frac{3}{5}-\frac{4}{5}=\frac{-2}{5} \qquad B=-\frac{1}{5}$

$$\frac{1}{x^3+5x^2+12x+8} = \frac{1}{5}\,\frac{1}{x+1} - \frac{1}{5}\,\frac{x}{(x+2)^2+4} - \frac{3}{5}\,\frac{1}{(x+2)^2+4}$$

$$= \frac{1}{5}\,\frac{1}{x+1} - \frac{1}{5}\,\frac{(x+2)}{(x+2)^2+4} + \frac{1}{5}\,\frac{1}{(x+2)^2+4}$$

$$\int \frac{dx}{x^3+5x^2+12x+8} = \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln|(x+2)^2+4| + \frac{1}{10}\arctan\left(\frac{x+2}{2}\right) + C$$

Lect 5 Practice

$$\int \frac{1+x}{x(x^4+2x^2+1)}\,dx = \int \frac{1+x}{x(x^2+1)^2}\,dx$$

$$\frac{1+x}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

$$1+x = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

Taking ~~a~~ $x=0$ gives

$1 = A.$

~~Taking~~

$$1+x = x^4+2x^2+1 \quad + Bx^4 + Cx^3 + Bx^2 + Cx \quad + Dx^2 + Ex$$

$x^4$ Terms $\quad 1+B=0 \quad B=-1$

$x^3$ Terms $\quad C=0$

$x^2$ Terms $\quad 2+B+D=0 \Rightarrow 1+D=0 \quad D=-1$

$x$ Terms $\quad 1 = \underbrace{\cancel{C}}_{0}x + E \quad E=1$

1 Terms $\quad 1 = 1$

$$\frac{1+x}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{(x^2+1)} - \frac{x}{(x^2+1)^2} + \frac{1}{(x^2+1)^2}$$

$$\int \frac{1+x}{x(x^2+1)^2} = \ln|x| - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}(x^2+1)^{-1}$$

$$+ \int \frac{dx}{(x^2+1)^2}$$

$\int \frac{dx}{(x^2+1)^2}$ can be done by Trig substitution

$x = \tan u$

$dx = \sec^2 u \, du$

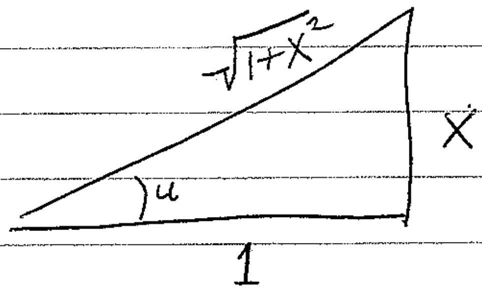
$$\int \frac{\sec^2 u \, du}{(1+\tan^2 u)^2} = \int \frac{\sec^2 u}{\sec^4 u} du = \int \sec^{-2} u \, du = \int \cos^2 u \, du$$

$$= \int \frac{1+\cos 2u}{2}$$

$$= \frac{u}{2} + \frac{\sin 2u}{4}$$

$$= \frac{u}{2} + \frac{1}{2}\sin u \cos u$$

Drawing a right Triangle




$$\tan u = \frac{X}{1} = \frac{o}{a}$$

$$\sin u = \frac{o}{H} = \frac{X}{\sqrt{1+X^2}}$$

$$\cos u = \frac{a}{H} = \frac{1}{\sqrt{1+X^2}}$$

$$\int \frac{dx}{(X^2+1)^2} = \frac{\arctan X}{2} + \frac{1}{2}\frac{X}{\sqrt{1+X^2}} \cdot \frac{1}{\sqrt{1+X^2}} + C$$

$$= \frac{1}{2}\arctan X + \frac{1}{2}\frac{X}{1+X^2} + C$$

$$\int \frac{1+X}{X(X^2+1)^2} = \ln|X| - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\frac{1}{1+X^2} + \frac{\arctan(x)}{2}$$

$$+ \frac{1}{2}\frac{X}{1+X^2} + C$$

$$\frac{2x^5+1}{(x-1)^3(x^4+2x^2+1)}$$

Note $x^4+2x^2+1 = (x^2+1)^2$

$$\frac{2x^5+1}{(x-1)^3(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{(x^2+1)} + \frac{Fx+G}{(x^2+1)^2}$$

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$$\frac{2}{(x+1)^3(x^2+1)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{(x^2+1)} + \frac{Fx+G}{(x^2+1)^2}$$

$$+ \frac{Hx+I}{(x^2+1)^3} + \frac{Jx+K}{(x^2+1)^4}$$

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Note: $x^4+6x^2+5 = (x^2+5)(x^2+1)$

$$\frac{5x-7}{x^4+6x^2+5} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+1)}$$

$$\lim_{x \to 0} x \ln^n x = \lim_{x \to 0} \frac{\ln^n x}{\frac{1}{x}}$$

by L'Hopital

$$(\ln^n x)' = \frac{n \ln^{n-1} x}{x}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\lim_{x \to 0} x \ln^n x = \lim_{x \to 0} \frac{-n \ln^{n-1} x}{\frac{1}{x}}$$

WHEN $n=1$

$$\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} -x = 0$$

by induction

$$\lim_{x \to 0} x \ln^2 x = -2 \lim_{x \to 0} x \ln x = 0$$

$$\lim_{x \to 0} x \ln^3 x = -3 \lim_{x \to 0} x \ln^2 x = 0$$

$$\vdots \quad \lim_{x \to 0} x \ln^n x = 0.$$

$$\lim_{R\to 0}\int_R^1 x^{-\alpha}\,dx = \left.\frac{x^{1-\alpha}}{1-\alpha}\right|_R^1 = \frac{1}{1-\alpha}\left(1 - R^{1-\alpha}\right)$$

as $R \to 0$ $\quad R^{1-\alpha} \to \begin{cases} 0 & \text{if } \alpha < 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$

For $\alpha = 1$

$$\int_R^1 x^{-1}\,dx = \ln|R| \to -\infty \quad \text{as } R \to 0$$

THUS $\displaystyle\int_0^1 x^{-\alpha}\,dx \begin{cases} \text{Diverges} & \alpha \geq 1 \\ \text{Converges} & \alpha < 1. \end{cases}$

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$$\int_0^{\frac{1}{2}} \frac{1}{x \ln^\beta x}\,dx = \int_{-\infty}^{-\ln 2} y^{-\beta}\,dy$$

$y = \ln x$

$dy = \frac{dx}{x}$

$$\int_{-R}^{-\ln 2} y^{-\beta}\,dy = \left.\frac{y^{1-\beta}}{1-\beta}\right|_{-R}^{-\ln 2} = \frac{(-\ln 2)^{1-\beta}}{1-\beta} - \frac{(-R)^{1-\beta}}{1-\beta}$$

as $R \to \infty$ $\quad R^{1-\beta} \to \begin{cases} 0 & \beta > 1 \\ \infty & \beta < 1 \end{cases}$

Converges $\beta > 1$ $\qquad$ Diverges $\beta \leq 1$.