Lecture 6 Practice Exercises: Improper Integrals

At what points do the following integrals become improper?

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$
$$\int_0^\infty x^3 e^{-x} dx$$
$$\int_{-1}^\infty \frac{dx}{\sqrt{|x|(x^3+x-2)}}$$
$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)dx}{\sqrt{\cos(x)}}$$
$$\int_0^5 xe^{-x} dx$$

Decide whether each of the above integrals converges or diverges.

Show that *for x large enough*

$$x^3 e^{-2x} \leq e^{-x}$$

(Give an estimate at how large x must be for the above inequality to hold.

Show that

$$\int_0^\infty x^3 e^{-2x} dx$$

converges using the comparison test.

LECTURE 6 Practice Problems

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$ improper @ $x=1$

Convergent!

$$\int_0^R \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)\Big|_0^R = \arcsin(R) - \arcsin(0) = \arcsin(R)$$

$$\lim_{R\to 1} \arcsin R = \arcsin(1) = \frac{\pi}{2}.$$

$$\int_0^\infty x^3 e^{-x}\,dx$$ improper @ $x=\infty$

Convergent.

integration by parts' gives

$$\int_0^R x^3 e^{-x}\,dx = -(x^3+3x^2+6x+6)e^{-x}\Big|_0^R$$

$$\lim_{R\to\infty} \int_0^R x^3 e^{-x} = 6 - \lim_{R\to\infty} (R^3+3R^2+6R+6)e^{-R}$$

$$= \underline{6}$$ Convergent

$$\int_{-1}^{\infty} \frac{dx}{\sqrt{x}\,(x^3+x-2)}$$

$$x^3+x-2 = (x-1)(x^2+x+2)$$

x^2+x+2 HAS NO REAL ROOTS,

$x-1$ vanishes @ $x=1$

$0, 1, \infty$

$$\int_{-1}^{0} \frac{dx}{x^{\frac{1}{2}}(x^3+x-2)} + \int_{0}^{\frac{1}{2}} \frac{dx}{x^{\frac{1}{2}}(x^3+x-2)} + \int_{\frac{1}{2}}^{1} \frac{dx}{x^{\frac{1}{2}}(x^3+x-2)}$$

$$+ \int_{1}^{2} \frac{dx}{x^{\frac{1}{2}}(x^3+x-2)} + \int_{2}^{\infty} \frac{dx}{x^{\frac{1}{2}}(x^3+x-2)}$$

DIVERGENT: BECAUSE of $x=1$

FOR $x \in 1, 2$ $\quad x^2+x-2 \leq 4 \quad x^{\frac{1}{2}} \leq \sqrt{2}$

$$\frac{1}{x^2+x-2} \geq \frac{1}{4} \qquad \frac{1}{x^{\frac{1}{2}}} \geq \frac{1}{\sqrt{2}}$$

$$\frac{1}{x^{\frac{1}{2}}(x-1)(x^2+x-2)} \geq \frac{1}{4\sqrt{2}} \frac{1}{(x-1)}$$

$$\int_1^2 \frac{dx}{x-1} \quad \underline{\text{divergent}}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \, dx}{\sqrt{\cos x}}$$ improper @ $x = \frac{\pi}{2}$

$$\int_0^R \frac{\sin x}{\sqrt{\cos x}}\, dx = -2(\cos x)^{\frac{1}{2}} \Big|_0^R$$

$$= 2 - 2(\cos R)^{\frac{c}{2}}$$

$$\lim_{R \to \frac{\pi}{2}} 2 - 2(\cos R)^{\frac{1}{2}} = \underline{2} \quad \underline{\text{Convergent}}$$

$$\int_0^5 x e^{-x}\, dx$$ <u>No problems</u> integrand continuous - <u>Convergent</u>

Show $x^3 e^{-2x} \le e^{-x}$ for x large enough.

This is equivalent to showing

$$x^3 e^{-x} \le 1 \quad \text{for } x \text{ large enough.}$$

Since $\lim_{x \to \infty} x^3 e^{-x} = 0$ it follows that for x large enough $x^3 e^{-x} < 1$

To estimate This:

$$f(x) = x^3 e^{-x}$$
$$f'(x) = (3x^2 - x^3)e^{-x}$$
$$= x^2(3-x)e^{-x}$$

so $f'(x)$ is <u>negative</u> for $x > 3$.

$f(5) = 5^3 e^{-5} \approx .84$

$f'(x) < 0$ for $x > 5$, so $f(x)$ is <u>decreasing</u>

<u>Thus</u>

$$x^3 e^{-2x} < e^{-x} \quad \text{for } \underline{x \geq 5}$$

$$\int_0^\infty x^3 e^{-2x}\,dx = \int_0^5 x^3 e^{-2x}\,dx + \int_5^\infty x^3 e^{-2x}\,dx$$

$$x^3 e^{-2x} \leq e^{-x} \quad \text{for } x \geq 5$$

Thus applying comparison Test w/ $f(x) = x^3 e^{-2x}$
$g(x) = e^{-x}$

shows $\int_5^\infty e^{-x}\,dx$ is convergent