

# 1 Sequences and Series

**Definition:** A sequence is a function whose domain is the set of integers: in other words for each integer we have a real number. Some examples:

**Example:**  $a_n = \frac{1}{n^2}$  or  $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \dots$

**Example:**  $a_n = \sin(\frac{n\pi}{4})$   $0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, \dots$

**Example:**  $a_n = 2 + \frac{-1}{n}$   $1, \frac{5}{2}, \frac{5}{3}, \frac{9}{4}, \frac{9}{5} \dots$

**Definition: Convergence**  $a_n \rightarrow L$  if for every  $\epsilon > 0$  there exists an  $N$  such that  $|a_n - L| < \epsilon$  for all  $n > N$

In other words, the game is this: In order for a sequence to converge the following has to hold: You give me a tolerance ( $\epsilon$ ) I have to find a number  $L$  (the limit) such that EVERY element after some point is within the tolerance ( $\epsilon$ ) of the limit ( $L$ ). If I can do this for EVERY tolerance  $\epsilon$  then the sequence is convergent.

**Examples:** Examples 1 and 3 above are convergent. Example 2 is not convergent.

**Proof**  $a_n = 1 - \frac{1}{n^2}$ . I claim that the limit is 1. In other words for every positive  $\epsilon$  I can find  $N$  such that  $n > N$  implies  $|a_n - 0| < \epsilon$ . I can choose  $N = \epsilon^{-1/2}$

Of course one basically NEVER evaluates a limit using the  $\epsilon, \delta$  definition. The following theorem is the one which is most useful:

**Theorem:** Suppose that  $\lim_{x \rightarrow \infty} f(x) = L$ . Then  $\lim_{n \rightarrow \infty} f(n) = L$

**Example:** Evaluate

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2n^2 + 1}{3n^2 + n + 18} \\ \lim_{x \rightarrow \infty} \frac{2x^2 + 1}{3x^2 + x + 18} &= \\ \lim_{x \rightarrow \infty} \frac{4x}{6x + 1} &= \\ \lim_{x \rightarrow \infty} \frac{4}{6} &= \frac{2}{3} \end{aligned}$$

By the above theorem  $\lim_{x \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} f(n) = L$ , so we are done.

**Example: Extended** Estimate how large we need to take  $N$  so that  $|a_n - \frac{2}{3}| \leq 10^{-3}$

**Note:** The converse is not true. Take  $f(x) = \sin(2\pi(x + \frac{1}{x}))$

Another useful result is the following:

**Theorem:** Suppose  $\{a_n\}_{n=1}^{\infty}$  converges, and  $\{b_n\}_{n=1}^{\infty}$  also converges. Then

- $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim a_n + \lim b_n$
- $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim a_n - \lim b_n$
- $\lim_{n \rightarrow \infty} (a_n b_n) = \lim a_n \times \lim b_n$
- $\lim_{n \rightarrow \infty} (a_n/b_n) = \frac{\lim a_n}{\lim b_n}$  if  $\lim b_n \neq 0$
- $\lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n)$  if  $f$  is continuous

and

**(Squeeze) Theorem:** Suppose  $a_n, b_n$  both converge to  $L$ , and there is an integer  $n_0$  such that for  $n > n_0$  we have  $a_n \leq c_n \leq b_n$ . The  $c_n$  also converges to  $L$ .

**Draw Picture!**

**Example:** Consider the sequence  $a_n = \frac{n \cos(n^2) + e^{-n}}{n^2}$ . Note that  $-n \leq n \cos(n^2) \leq n$ . Thus we have that  $\frac{1-n}{n^2} \leq a_n \leq \frac{n+1}{n^2}$ . Note that, by the L'Hopital's rule  $\frac{1-n}{n^2} \rightarrow 0$  and  $\frac{n+1}{n^2} \rightarrow 0$ . Thus by the squeezing theorem  $a_n \rightarrow 0$

**Corollary** If  $|a_n| \rightarrow 0$  then  $a_n \rightarrow 0$  **Pf:** Observe that  $-|a_n| \leq a_n \leq |a_n|$

**Example**  $a_n = \frac{(-1)^n}{n}$  converges to 0

**Flannery O'Connor Theorem:** Everything that rises must converge

If a sequence is increasing and bounded above or decreasing and bounded below it converges.

**Example:**  $a_n = \frac{2n+5}{n+1}$ . Computing  $a_{n+1} - a_n = \frac{-3}{(n+1)(n+2)}$  so  $a_n$  is decreasing.  $a_n$  is bounded below. Thus  $a_n$  converges.

Example:  $a_n = \frac{n!2^n}{n^{n-1}}$  converges to zero.

**Infinite Series:** A concept related to the idea of a sequence is an (infinite) series. A series is an infinite sum of the form

$$\sum_{n=1}^{\infty} a_n$$

We say that a series converges if the limit

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N a_n$$

exists (This is the sequence of partial sums). Note that this is **exactly** analogous to improper integrals, where we say that the integral converges if the limit

$$\lim_{R \rightarrow \infty} \int_1^R f(x) dx$$

converges.

The canonical example of an infinite series is a decimal representation: for instance

$$\pi = 3.14159\dots$$

which really means

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \dots$$

Note that if we are trying to represent **any real number** which is not a fraction of the form

$$\frac{k}{2^a 5^b}$$

then the decimal necessarily continues indefinitely.

It is actually EASY to see that decimals always converge. Note that the sequence 3, 3.1, 3.14, 3.141, 3.1415, ... is (i) *Increasing* and (ii) *Bounded* (since it is less than 3.2, say). So by the Flannery O'Connor Theorem (Theorem 1.4) the sequence must converge.

### Geometric Series

Let us begin with a simple example: the series for  $\frac{1}{9} = .111111\dots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$ . Given a repeating decimal we learn in junior high the algorithm for computing the fraction corresponding to a repeating decimal.

$$S = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \tag{1}$$

$$\frac{1}{10}S = \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \dots \tag{2}$$

$$\frac{9}{10}S = \frac{1}{10} \tag{3}$$

$$S = \frac{1}{9} \tag{4}$$

**A cautionary example:** The above calculation does **NOT** by itself constitute a proof that the series converges. Consider the following example:

$$S = \sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + 32 + \dots$$

The same argument gives

$$\begin{aligned} S &= \sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + 32 + \dots \\ 2S &= \sum_{n=1}^{\infty} 2^n = 4 + 8 + 16 + 32 + \dots \\ -S &= 2 \end{aligned}$$

Which is obviously **Nonsense!** So actually summing the series is not enough - one must really check for convergence. However this method does suggest a method for **proving** convergence

$$\begin{aligned} S_N &= \sum_{n=1}^N \frac{1}{10^n} = \frac{1}{10} + \frac{1}{100} + \dots + \frac{1}{10^N} \\ \frac{1}{10} S_N &= \frac{1}{100} + \dots + \frac{1}{10^N} + \frac{1}{10^{N+1}} \end{aligned}$$

So we have

$$\frac{9}{10} S_N = \frac{1}{10} - \frac{1}{10^{N+1}}$$

or

$$S_N = \frac{1}{9} \left( 1 - \frac{1}{10^N} \right)$$

Since it is clear that  $\frac{1}{10^N} \rightarrow 0$  as  $N \rightarrow \infty$  we have  $S_N \rightarrow \frac{1}{9}$ .

If we do the cautionary example more carefully, we find something **very** different:

$$\begin{aligned} S_N &= \sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + 32 + \dots + 2^N \\ 2S_N &= \sum_{n=1}^{\infty} 2^n = 4 + 8 + 16 + 32 + \dots + 2^N + 2^{N+1} \\ -S_N &= 2 - 2^{N+1} \\ S_N &= 2^{N+1} - 2 \end{aligned}$$

It is clear that the  $S_N$  diverges! When we do the above manipulations we are **assuming that the series converges**. If this is NOT the case then the answer we get is meaningless.

So it is important to test for convergence. Let us recall some facts about geometric series:

**Example:** Geometric Series

$$\sum_{k=0}^{\infty} ar^k$$

converges if  $|r| < 1$ . It diverges if  $|r| \geq 1$ .

**Example:** Investigate the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$$

Note that

$$\begin{aligned} S_1 &= 1/3 \\ s_5 &= 0.595 \\ S_{100} &= .740 \\ S_{1000} &= .749 \\ S_{10000} &= .7499 \\ S_{1000000} &= 0.749999 \end{aligned}$$