Problem 1:

$$\int x \sin^2(x)dx$$

Easiest Method: Integrate by parts

$$\int x \sin^2 x \, dx = \int x \frac{(1-\cos 2x)}{2} dx = \frac{x^2}{4} - \int \frac{x \cos 2x}{2} dx$$

Identity:

$$\cos^2 x - \sin^2 x = \cos 2x$$
$$\cos^2 x + \sin^2 x = 1$$
$$2\sin^2 x = 1 - \cos 2x$$

$$\int \frac{x \cos 2x}{2} dx = \frac{x \sin 2x}{4} - \int \frac{\sin 2x}{4} dx$$

$u = x \quad dV = \frac{\cos 2x}{2} dx$

$du = dx \quad V = \frac{\sin 2x}{4}$

$$= \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C$$

Thus

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + C$$

Problem 2:

$$\int \frac{dx}{x(x^2+1)}$$

Easiest: partial fractions

$$\int \frac{dx}{x(x^2+1)}$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$$

multiplying through by denominator gives

$$1 = A(x^2+1) + (Bx+C)x$$

Comparing terms

$$\left.\begin{array}{lll} x^2 \text{ Terms} & & A+B=0 \\ x \text{ Terms} & & C=0 \\ \text{constant Terms} & & A=1 \end{array}\right\} \quad \text{Thus } \begin{array}{l} A=1 \\ B=-1 \\ C=0 \end{array}$$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\int \frac{dx}{x(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln|x| - \tfrac{1}{2}\ln|1+x^2| + C$$

METHOD 2: Trig SUBSTITUTION

$$\int \frac{dx}{x(x^2+1)} = \int \frac{\sec^2 u\, du}{\tan u\,(1+\tan^2 u)}$$

$x = \tan u$

$dx = \sec^2 u\, du$

$1+\tan^2 u = \sec^2 u$

$$\int \frac{\sec^2 u\, du}{\tan u\,(1+\tan^2 u)} = \int \frac{du}{\tan u} = \int \frac{\cos u\, du}{\sin u}$$

simple sub: $y = \sin u$

$dy = \cos u\, du$

$$\int \frac{\cos u\, du}{\sin u} = \int \frac{dy}{y} = \ln|y| + C$$

$$= \ln|\sin u| + C$$

MUST solve for $\sin u$ in Terms of x

$$x = \frac{\sin u}{\cos u} \qquad x^2 = \frac{\sin^2 u}{\cos^2 u}$$

$$\frac{1}{x^2} = \frac{\cos^2 u}{\sin^2 u} \qquad 1+\frac{1}{x^2} = 1+\frac{\cos^2 u}{\sin^2 u} = \frac{\sin^2 + \cos^2}{\sin^2 u}$$

$$= \frac{1}{\sin^2 u}$$

$$1+\frac{1}{x^2} = \frac{1}{\sin^2 u} \qquad \sin u = \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{x(x^2+1)} = \ln \frac{x}{\sqrt{1+x^2}} + C = \ln x - \frac{1}{2}\ln|1+x^2| + C$$

Problem 3:

$$\int \sec^2(x)\tan^2(x)dx$$

Trig substitution

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int \underbrace{\sec^2 x \overbrace{\tan^2 x}^{u^2} dx}_{du} = \int u^2 du = \frac{u^3}{3} + C$$

$$= \frac{\tan^3 x}{3} + C$$

Problem 4:

$$\int \frac{dx}{x(x^2-1)^{\frac{3}{2}}}$$

Trig substitution

$$x = \sec u$$
$$dx = \sec u \tan u \, du$$

Identities
$$\begin{cases} 1+\tan^2 = \sec^2 \\ \sec^2 u - 1 = \tan^2 u \end{cases}$$

$$\int \frac{dx}{x(x^2-1)^{3/2}} = \int \frac{\sec u \tan u \, du}{\sec u \; (\sec^2 u - 1)^{3/2}} = \int \frac{\tan u}{(\tan^2 u)^{3/2}} \, du$$
$$= \int \frac{du}{\tan^2 u}$$

2ND TRIG SUBSTITUTION! $Q = \tan u$, $dQ = \sec^2 u \, du$ $\quad du = \frac{dQ}{\sec^2 u} = \frac{dQ}{1+\tan^2 u} = \frac{dQ}{1+Q^2}$

$$\int \frac{du}{\tan^2 u} = \int \frac{dQ}{Q^2(1+Q^2)} = -\frac{1}{Q} - \arctan(Q) + C$$

PARTIAL fractions $\frac{1}{Q^2(1+Q^2)} = \frac{1}{Q^2} - \frac{1}{(1+Q^2)}$

$$= -\cot u - u + C$$

$$x^2 = \sec^2 u = \frac{1}{\cos^2 u}$$
$$x^2 - 1 = \frac{1-\cos^2 u}{\cos^2 u} = \frac{\sin^2 u}{\cos^2 u} = \tan^2 u \quad \Rightarrow \cot u = \frac{1}{\sqrt{x^2-1}}$$

$$= -\frac{1}{\sqrt{x^2-1}} - \operatorname{arcsec}(x) + C$$

Simpler METHOD (BUT Tricky!)

$$\int \frac{dx}{x(x^2-1)^{3/2}}$$

NOTICE: $$\frac{1}{(x^2-1)^{3/2}} = \frac{x^2}{(x^2-1)^{3/2}} - \frac{1}{(x^2-1)^{1/2}}$$

$$\int \frac{dx}{x(x^2-1)^{3/2}} = \int \frac{1}{x}\left(\frac{x^2}{(x^2-1)^{3/2}} - \frac{1}{(x^2-1)^{\frac{1}{2}}}\right) dx$$

$$= \int \frac{x}{(x^2-1)^{3/2}}\, dx - \int \frac{dx}{x(x^2-1)^{\frac{1}{2}}}$$

simple substitution

$u = x^2$

$du = 2x\,dx$

$$= -(x^2-1)^{-\frac{1}{2}} - \operatorname{arcsec}(x) + C$$

Problem 5:

$$\int \frac{dx}{x(x-1)}$$

Straight forward partial fraction:

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$A(x-1) + B(x) = 1$$

$$A + B = 0$$

$$-A = 1$$

$$A = -1 \quad B = 1$$

$$\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$$

$$\int \frac{dx}{x(x-1)} = \int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = \ln(x-1) - \ln x + C$$

State the comparison test for improper integrals, and determine whether or not the following integral converges by comparing to an appropriate test function:

$$\int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$$

Note That $X+1 > X$

Thus $\dfrac{1}{\sqrt{X}(X+1)} \leq \dfrac{1}{\sqrt{X}\,X}$

The comparison Test STATES:

"Suppose $0 \leq f(x) \leq g(x)$. Then The following are True.

(i) If $\int g(x)\,dx$ <u>CONVERGES</u> Then $\int f(x)\,dx$ <u>converge</u>

(ii) If $\int f(x)\,dx$ diVERGES Then $\int g(x)\,dx$ <u>diverges</u>

NOTE: <u>CONVERGENCE</u> of $\int f(x)\,dx$ OR <u>DIVERGENCE</u> of $\int g(x)\,dx$ TELL US <u>NOTHING</u>

SO WE TAKE $g(x) = \dfrac{1}{\sqrt{X}\,X} = X^{-3/2}$ $\quad f(x) = \dfrac{1}{\sqrt{X}(X+1)}$

$$\lim_{R\to\infty} \int_1^R \frac{dx}{\sqrt{X}\,\cancel{(X+1)}} = \lim_{R\to\infty} \int_1^R X^{-3/2} = \frac{X^{-\frac{1}{2}}}{-\frac{1}{2}}\Big|_1^R \underset{\lim_{R\to\infty}}{=} 2 - 2R^{-\frac{1}{2}}$$

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$$\lim_{R\to\infty} \int_1^R X^{-\frac{3}{2}}\,dx = 2 \quad \underline{\text{CONVERGENT}}$$

<u>SO</u> $\displaystyle\int \frac{dx}{\sqrt{X}(X+1)}$ <u>converges</u>