Math 231 Practice Exam 2 SOLUTIONS

Problem 1:

State as briefly and concisely as you can the following theorems/tests. List all hypothesis and the conclusions. Your statement should be formulated "If (hypothesis) Then (conclusions)." If it is possible for the test to fail so indicate.

(i)**Integral Test**

(**Hypothesis**) *Suppose that* $f(x)$ *is a positive, decreasing function and* $a_n = f(n)$*.* then (**Conclusion**) the series $\sum a_n$ and the (improper)integral $\int_1^\infty f(x)dx$ either both converge or both diverge.

(ii) **Limit Comparison Test**

Hypothesis*Suppose that* $a_n > 0$ *for all* n *and* $\lim \frac{a_n}{b_n} = L > 0$ *Then* **(Conclusion)** *the series* $\sum a_n$ *and* $\sum b_n$ *either both converge or they both diverge.*

(iii)**Comparison Test for SEQUENCES**

Hypothesis*Suppose that* $0 < a_n \leq b_n$ *for all* n *(sufficiently large) Then* **(Conclusions)**

- $\sum b_n$ converges implies *the series* $\sum a_n$ *converges*
- $\sum a_n$ diverges implies *the series* $\sum b_n$ *diverges*

(iv)k^{th} **Term Test** There are two ways to state this: the way the test is stated in the book or the contrapositive, which is how it is usually used

*(**Hypothesis**) Suppose* $\sum a_n$ *converges. then (**Conclusion**)* $\lim_{n\to\infty} a_n = 0$

The logical opposite (negation) of this is

*(**Hypothesis**) Suppose* $\lim_{n\to\infty} a_n \neq 0$ *then (**Conclusion**)* $\sum a_n$ *does not converge.*

COMMON MISTAKE: Many students make the mistake of assuming that the logical opposite of the k^{th} term test is *(**Hypothesis**) Suppose* $\lim_{n\to\infty} a_n = 0$ *then (**Conclusion**)* $\sum a_n$ *converges.*

THIS IS NOT TRUE! The harmonic series provides a counterexample to this, as does $\sum n^{-p}$ for all $p \leq 1$. **DO NOT MAKE THIS MISTAKE!**

Problem 2: Short Answer:

(i) A sequence is decreasing ($a_{n+1} < a_n$) and bounded above ($a_n < 1$ for all n). Is it necessarily true that the sequence converges? **Either** show that the sequence must converge **or** give an example which does not converge.

NOT TRUE: *Consider the sequence* $a_n = -n$. *This is decreasing and does not converge.*

(ii) A sequence is increasing ($a_{n+1} > a_n$) and bounded above ($a_n < 1$ for all n). Is it necessarily true that the sequence converges? **Either** show that the sequence must converge **or** give an example which does not converge.

TRUE: *There is a theorem that says that every increasing sequence which is bounded above has a limit.*

(iii) Define what it means for a sequence to converge to a limit L.

A sequence a_n converges to L if for every $\epsilon > 0$ there exists a number N such that for every $n > N$ we have $|a_n - L| < \epsilon$

In other words: if you give me any positive distance (ϵ) then every term after some point (N) is within that distance of the limit L

(iv) Can a sequence a_n converge to two different limits? Either give an example of a sequence converging to two different limits **or** give a convincing argument why this is not possible.

NO! *If a sequence converged to two limits L_1,L_2 then every term is eventually within ϵ of both L_1 and L_2. If ϵ is much smaller than the distance between L_1 and L_2 this is not possible. You can't be within 50 miles of New York and within 50 miles of Chicago at the same time!*

Problem 3: Assess the convergence or divergence of the following sequence. If it converges evaluate the limit

$$a_n = \frac{n}{n+1} + (-1)^n \frac{n^2 - 1}{n^2 + 1}$$

DIVERGES *Note that odd terms approach* 0 *and even terms approach* 2*. A sequence can't have two limits.*

Problem 4: Assess the convergence or divergence of the following series

$$\sum_{n=1}^{\infty} \frac{n-1}{\sqrt{n^{4.1} + 1}}$$

CONVERGES *Easiest: limit comparison test* $b_n = \frac{1}{n^{3.1}}$

Problem 5:
Assess the convergence or divergence of the series

$$\sum_{n=3}^{\infty} \frac{1}{(n-1)\log(n)}$$

Diverges: *Easiest: limit comparison test + integral test By the limit comparison test this has the same behavior (convergence or divergence) as*

$$\sum_{n=3}^{\infty} \frac{1}{(n)\log(n)}$$

By the integral test this has the same behavior as

$$\int n = 3^{\infty} \frac{dx}{x\log(x)}$$

This can be done by the simple substitution $u = \ln(x)$ *giving*

$$\int n = 3^{R} \frac{dx}{x\log(x)} = \ln(\ln(R))$$

which diverges as $R \to \infty$

Problem 6: Assess the convergence or divergence of the series

$$\sum_{n=3}^{\infty} \frac{1}{n^{2008} + 5001\log(n) + \cos^2(n)}$$

Converges: *Limit comparison test with* $b_n = \frac{1}{n^{2008}}$

Problem 7: Assess the convergence or divergence of the sequence

$$a_n = \frac{n^{2008}}{e^n + n^{2008}}$$

CONVERGES to 0 e^n *grows faster than any power of* n. *Applying L'Hopital's rule* 2009 *times will give* 0 *in the numerator and* e^n *in the denominator. Thus the limit is* 0

If the sequence converges evaluate the limit.

Problem 8: Does the following series converge? If so evaluate the sum

$$\sum \frac{1}{k(k+3)}$$

Converges: *By limit comparison test to* $\sum k^{-2}$. *Using the partial fractions expansion we get*

$$\frac{1}{k(k+3)} = \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right)$$

Writing out thefirst few terms of the series gives

$$\sum_{k=1} \frac{1}{k(k+3)} = \frac{1}{3}\left(1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{8} + \dots\right)$$

Noting that all but the first three terms cancel gives

$$\sum_{k=1} \frac{1}{k(k+3)} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}$$