Math 231 Practice Exam 3

Instructions: This is a practice exam. Please treat it as a regular exam: sit down and take it in 50 minutes without interruption and without reference to the textbook or to the class notes. After this you may want to spend some time going through it carefully to see how you did. The TAs will hand out solutions in sections on Tuesday. When answering questions on the convergence or divergence of a sequence or series you **MUST** give a proof or cite an appropriate theorem or theorems.

Problem 1:

State as briefly and concisely as you can the following theorems/tests. List all hypothesis and the conclusions. Your statement should be formulated "If (hypothesis) Then (conclusions)." If it is possible for the test to fail so indicate.

(i)**Ratio Test**

Solution:

Given the series $\sum a_k$, suppose that the limit $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = R$.

- If $R < 1$ then the series converges absolutely.
- If $R > 1$ then the series diverges.
- If $R = 1$ then the test is inconclusive - the series may converge or diverge.

(ii) **Taylor Series Formula**

Solution:

$$f(x) = \sum_{k=0}^{\infty} \frac{d^k f}{dx^k}(c)\frac{(x-c)^k}{k!}$$

The error estimate for $P_N(x)$ the Taylor polynomial of degree N, is

$$P_N(x) - f(x) = \frac{d^{N+1} f}{dx^{N+1}}(z)\frac{(x-c)^{N+1}}{(N+1)!}$$

for some point $z \in (c, x)$

(iii) **Root Test**

Solution:

Given the series $\sum a_k$, suppose that the limit $\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = R$.

- If $R < 1$ then the series converges absolutely.
- If $R > 1$ then the series diverges.
- If $R = 1$ then the test is inconclusive - the series may converge or diverge.

Problem 2: Compute the radii of convergence of the following power series:
(i)

$$\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k$$

SOLUTION: Using the ratio test we have

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \frac{k!}{k^k} \\ &= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)k!} \frac{k!}{k^k} \\ &= \lim_{k\to\infty} \frac{(k+1)^k}{k^k} \\ &= \lim_{k\to\infty} (1+\frac{1}{k})^k \end{aligned}$$

by the compound interest formula $\lim_{k\to\infty}(1+\frac{1}{k})^k = e$ Thus the radius of convergence is $\frac{1}{e}$.

(ii)

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3+2^k} x^k$$

SOLUTION: Using the ratio test we have

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)^2}{(k+1)^3+2^{k+1}} \frac{k^3+2^k}{k^2}$$

Recall that **IF** $\lim_{k\to\infty} a_k = A$ and $\lim_{k\to\infty} b_k = B$ both exist then $\lim_{k\to\infty} a_k b_k$ exists and is equal to AB. By L'Hopital

$$\lim_{k\to\infty} \frac{(k+1)^2}{k^2} = \lim_{k\to\infty} \frac{2(k+1)}{2k} = \lim_{k\to\infty} \frac{2}{2}$$

and similarly

$$\lim_{k\to\infty} \frac{(k)^3+2^k}{(k+1)^3+2^{k+1}} = \lim_{k\to\infty} \frac{3k^2+\ln(2)2^k}{3(k+1)^2+\ln(2)2^{k+1}} = \lim_{k\to\infty} \frac{6(k)+\ln^2(2)2^k}{6(k+1)+\ln^2(2)2^{k+1}} = \lim_{k\to\infty} \frac{6+\ln^3(2)2^k}{6+\ln^3(2)2^{k+1}} =$$

Thus we have

$$\lim_{k\to\infty} \frac{(k+1)^2}{(k+1)^3+2^{k+1}} \frac{k^3+2^k}{k^2} = \left(\lim_{k\to\infty} \frac{(k+1)^2}{k^2}\right) \left(\lim_{k\to\infty} \frac{(k)^3+2^k}{(k+1)^3+2^{k+1}}\right) = \frac{1}{2} \times 1 = \frac{1}{2}$$

and the radius of convergence is thus $\frac{1}{\frac{1}{2}} = 2$.

(iii)

$$\sum_{k=1}^{\infty} \frac{k^2}{k!} x^k$$

SOLUTION:

$$\lim_{k\to\infty} \frac{(k+1)^2}{(k+1)!} \frac{k!}{k^2} = \lim_{k\to\infty} \frac{(k+1)}{k^2} = 0$$

where the latter follows from applying L'hopital twice (giving a numerator of 0 and a denominator of 2. Thus the radius of convergence is infinite.

Problem 3:

Compute the Taylor series for the given function about the given point: Give at least three terms. If you can guess the general form.

(i) $f(x) = \sin(x) \quad c = 1$

Solution

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \end{aligned}$$

Thus the Taylor series begins

$$\sin(x) = \sin(1) + \cos(1)(x-1) - \sin(1)\frac{(x-1)^2}{2!} - \cos(1)\frac{(x-1)^3}{3!} + \dots$$

The closed form expression is

$$\sin(x) = \sin(1)\left(1 - \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} - \frac{(x-1)^6}{6!} + \dots\right) + \cos(1)\left(x - \frac{(x-1)^3}{3!} + \frac{(x-1)^5}{5!} - \frac{(x-1)^7}{7!} + \right.$$

(ii) $f(x) = \tan(x) \quad c = 0$

Solution:

$$\begin{aligned} f'(x) &= \sec^2(x) \\ f''(x) &= 2\sec^2(x)\tan(x) \\ f'''(x) &= 2\sec^4(x) + 4\sec^2(x)\tan^2(x) \end{aligned}$$

Since $\sec(0) = 1$ and $\tan(0) = 0$ we have

$$f(x) = x + 2\frac{x^3}{6!} + \dots = x + \frac{x^3}{3} + \dots$$

Fact: The terms of the Taylor series for the Tangent can be expressed in terms of something called the Bernoulli numbers.

(iii) $f(x) = e^{-x^2} \quad c = 0$

Solution: This follows from the Taylor series for the exponential:

$$f(x) = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

Problem 4:

All of the following series have radius of convergence equal to 1. Decide whether the series converge on the boundaries $|x - c| = \pm 1$

Important Observation: Since one finds the radius of convergence from the ratio (or root) test, and the test is inconclusive at the boundary you are always going to want to try a more sensitive test to determine the convergence at the boundary, usually limit, limit comparison, or integral tests.

(i)

$$\sum_{k=0}^{\infty} \frac{x^k}{2k+1}$$

Solution: We're already told that the radius of convergence is 1. At $x = 1$ we get the series

$$\sum \frac{1}{2k+1}$$

This looks a bit like the harmonic series, except with a $2k + 1$ instead of a k. So we apply the limit comparison test with $a_k = \frac{1}{2k+1}$ and $b_k = \frac{1}{k}$. It is easy to see by L'Hopital that

$$\lim_{k \to \infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$$

Thus EITHER both series converge or both DIVERGE. Since we know that the harmonic series diverges both series must diverge.

At $x = -1$ we have the series

$$\sum (-1)^k \frac{1}{2k+1}$$

The coefficients $a_k = \frac{1}{2k+1}$ are positive, decreasing, and tend to zero. Thus by the alternating series test the series converges.

(ii)

$$\sum_{k=0}^{\infty} \frac{(x-5)^k}{k^3+11}$$

The radius of convergence is 1, so we must check the points $x = 6$ and $x = 4$. At $x = 6$ we get

$$\sum_{k=0}^{\infty} \frac{1}{k^3+11}$$

Applying the limit comparison test with $\sum \frac{1}{k^3}$ shows that this converges. At $x = 4$ we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k^3+11}$$

It is easy to see that $\sum |a_k| = \sum \frac{1}{k^3+11}$ is the series above, which is convergent. So the above series is absolutely convergent, and thus convergent.

Problem 5:
Evaluate the following limits using Taylor series: (i)

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^3}$$

Solution:
The Taylor series for e^{x^2} follows from the exponential series by substitution:

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots$$

so we have

$$e^{x^2} - 1 - x^2 = \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots$$

and

$$\frac{e^{x^2} - 1 - x^2}{x^3} = \frac{x}{2!} + \frac{x^3}{3!} + \frac{x^5}{4!} + \dots$$

Letting $x \to 0$ gives

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^3} = 0$$

Since the righthand side vanishes at $x = 0$.

(ii)

$$\lim_{x \to 0} \frac{\sin(x^2) - x^2}{x^4}$$

Solution Similarly to the above we have

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \dots$$

and

$$\frac{\sin(x^2) - x^2}{x^4} = -\frac{x^2}{3!} + \frac{x^6}{5!} - \frac{x^{10}}{7!} + \dots$$

Allowing $x \to 0$ gives

$$\lim_{x \to 0} \frac{\sin(x^2) - x^2}{x^4} = 0$$

(iii)

$$\lim_{x \to 0} \frac{\cos(2x) - 1 - 2x^2}{x^6}$$

Solution There is a minor typo in this - the numerator should have read $\cos(2x) - 1 + 2x^2$, but it doesn't really change anything. Again we have

$$\cos(2x) - 1 - 2x^2 = -4x^2 + \frac{2}{3}x^4 - \frac{2^6}{6!}x^6 + \frac{2^8}{8!}x^8 + \dots$$

Dividing by x^6 gives

$$\frac{\cos(2x) - 1 - 2x^2}{x^6} = \frac{-4x^2 + \frac{2}{3}x^4}{x^6} - \frac{4}{45} - \frac{2^6}{6!} + \frac{2^8}{8!}x^2 + \dots$$

As $x \to 0$ we have to evaluate the limit

$$\lim_{x \to 0} \frac{-4x^2 + \frac{2}{3}x^4}{x^6} - \frac{4}{45}$$

It is easy to see using L'Hopital that this diverges. Thus the limit does not exist.