

Math 231 Practice Final

Instructions: This is a practice exam. Please treat it as a regular exam: sit down and take it in 3 hours without interruption and without reference to the textbook or to the class notes. After this you may want to spend some time going through it carefully to see how you did. The TAs will hand out solutions in sections on Tuesday. When answering questions on the convergence or divergence of a sequence or series you **MUST** give a proof or cite an appropriate theorem or theorems.

Solutions

Problem 1:

Evaluate the following indefinite integrals

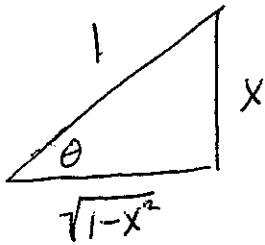
(i)

$$\int \frac{x^3}{\sqrt{1-x^2}} dx$$

Trig sub

$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$



$$\sin \theta = \frac{x}{1} = x$$

$$\cos \theta = \sqrt{1-x^2}$$

$$\int \frac{\sin^3 \theta (\cos \theta d\theta)}{\sqrt{1-\sin^2 \theta}} = \int \sin^3 \theta d\theta$$

$$= \int \sin \theta (1-\cos^2 \theta) d\theta$$

$$u = \cos \theta$$

$$du = -\sin \theta d\theta$$

$$= -\int (1-u^2) du$$

$$= -u + \frac{u^3}{3} + C$$

$$= -\cos \theta + \frac{\cos^3 \theta}{3} + C$$

$$= \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2} + C$$

(ii)

$$\int \sin^2(x) e^{3x} dx$$

$$I = \int \frac{(1-\cos 2x)}{2} e^{3x} dx \quad (\sin^2 x = \frac{1}{2}(1-\cos 2x))$$

$$= \frac{1}{6} e^{3x} - \frac{1}{2} \int \cos 2x e^{3x} dx$$

$$\text{Let } I_2 = \int \cos 2x e^{3x} dx$$

$$Q = \cos 2x \quad dR = e^{3x} dx$$

$$dQ = -2\sin 2x dx \quad R = \frac{e^{3x}}{3}$$

$$I_2 = \frac{1}{3} \cos 2x e^{3x} + \frac{2}{3} \int \sin 2x e^{3x} dx$$

$$Q = \sin 2x \quad dR = e^{3x} dx$$

$$dQ = 2\cos 2x dx \quad R = \frac{e^{3x}}{3}$$

$$I_2 = \frac{1}{3} \cos 2x e^{3x} + \frac{2}{9} \sin 2x e^{3x} - \frac{4}{9} \int \cos 2x e^{3x} dx$$

$$I_2 = \frac{1}{3} \cos 2x e^{3x} + \frac{2}{9} \sin 2x e^{3x} - \frac{4}{9} I_2$$

$$\frac{13}{9} I_2 = \frac{1}{3} \cos 2x e^{3x} + \frac{2}{9} \sin 2x e^{3x}$$

$$I_2 = \frac{3}{13} \cos 2x e^{3x} + \frac{2}{13} \sin 2x e^{3x}$$

Thus $I = \frac{1}{6} e^{3x}$
 $-\frac{1}{2} \left(\frac{3}{13} \cos 2x e^{3x} + \frac{2}{13} \sin 2x e^{3x} \right)$

Sym. Division

$$\begin{array}{r}
 X^2 + X^{(iii)} + 1 \\
 X-1 \overline{) X^3 + 0X^2 + 0X - 1} \\
 \underline{X^3 - X^2} \\
 X^2 \\
 \underline{X^2 - X} \\
 X - 1
 \end{array}$$

$$\int \frac{1}{x^3-1} dx$$

$X-1$ is a FACTOR of X^3-1

$$X^3-1 = (X-1)\left(X+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\frac{1}{X^3-1} = \frac{A}{X-1} + \frac{Bx+C}{\left(X+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$A(X^2+X+1) + BX(X-1) + C(X-1) = 1$$

$$A+B=0 \quad A+B=0$$

$$A-B+C=0 \quad 2A-B=1$$

$$A-C=1$$

(iv)

$$\begin{aligned}
 A &= \frac{1}{3} \\
 B &= -\frac{1}{3} \\
 C &= A-1 = -\frac{2}{3}
 \end{aligned}$$

$$= \int \frac{1}{3} \frac{1}{x-1} + \int \frac{-\frac{1}{3}x}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{2}{3} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\begin{aligned}
 \int x \arctan(x) dx &= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln\left(\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right) \\
 &\quad - \frac{2}{\sqrt{3}} \operatorname{Arctan}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right)
 \end{aligned}$$

$$\int x \arctan x dx$$

$$dQ = x dx \quad R = \arctan(x)$$

$$Q = \frac{x^2}{2} \quad dR = \frac{dx}{1+x^2}$$

$$= \frac{x^2}{2} \arctan(x) - \int \frac{x^2}{2} \frac{dx}{1+x^2}$$

$$= \frac{x^2}{2} \arctan(x) - \frac{1}{2} \int \frac{(x^2+1-1)}{1+x^2} dx$$

$$= \frac{x^2}{2} \arctan(x) - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan(x) + C$$

(v)

$$\int \frac{1}{x^2(x-1)^2} dx = \int \left(\frac{1}{x^2} + \frac{2}{x} + \frac{1}{(x-1)^2} - \frac{2}{x-1} \right) dx$$

Partial fractions:

$$\frac{1}{x^2(x-1)^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

$$A(x^2-2x+1) + Bx(x^2-2x+1) + Cx^2 + Dx^2(x-1) = 1$$

$$x^3 \quad B + D = 0$$

$$x^2 \quad A - 2B + C - D = 0$$

$$x \quad -2A + B = 0$$

$$1 \quad A = 1 \quad B = C = 1$$

$$\Rightarrow B = 2 \quad D = -2 \quad 1 - 4 + C + 2 = 0$$

(iv)

$$\text{Recall: } \sec' x = \sec x \tan x$$

$$\tan' x = \sec^2 x$$

$$1 + \tan^2 x = \sec^2 x$$

$$\int \sec(x) \tan^2(x) dx = \sec x \tan x - \int \sec^3 x$$

$$Q = \sec x \tan x \quad R = \tan x$$

$$Q' = \sec x \quad dR = \sec^2 x dx$$

$$\text{NOTE: } \int \sec^3 x = \int \sec(1 + \tan^2 x) = \int \sec dx + \int \sec \tan^2 x$$

$$\text{LETTING } I = \int \sec \tan^2 x \text{ gives}$$

$$I = \sec x \tan x + \int \sec dx - I$$

$$I = \frac{1}{2} \sec x \tan x - \frac{1}{2} \int \sec x dx$$

$$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \int \sec x (\sec x + \tan x) dx$$

$$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln(\sec x + \tan x) + C$$

$$\int \cos^2(x) \sin(2x) dx$$

(vi)

$$\sin 2x = 2 \sin x \cos x$$

$$\int \cos^2 x \sin 2x = 2 \int \sin x \cos^3 x dx = -2 \int u^3 du$$

$$u = \cos x \quad du = -\sin x dx = -\frac{u^4}{2} + C$$

$$= -\frac{\cos^4 x}{2} + C$$

See Text Example

Problem 2: Compute the radii of convergence of the following power series.
If the series has a finite radius of convergence check the endpoints to determine if the series converges there.

(i)

$$\sum_{k=1}^{\infty} \frac{k!}{k^{2k}} x^k$$

RATIO TEST

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{(2k+2)}} \cdot \frac{k^{2k} x^{k+1}}{(k!) x^k} = \frac{(k+1) k! k^{2k}}{(k+1)^{2k} (k+1)^2 k!} = \frac{1}{k+1} \left(\frac{k}{k+1}\right)^{2k} x$$

Note $\frac{k}{k+1} < 1 \Rightarrow \left(\frac{k}{k+1}\right)^{2k} < 1 \Rightarrow \left(\frac{k}{k+1}\right)^{2k} \frac{1}{k+1} < \frac{1}{k+1}$

$\lim_{k \rightarrow \infty} \frac{1}{k+1} = 0$ Thus $\lim_{k \rightarrow \infty} \left(\frac{k}{k+1}\right)^{2k} \frac{1}{k+1} = 0$.

Converges all x .

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3 + \sin(k)} x^{2k}$$

RATIO TEST

$$\lim_{k \rightarrow \infty} \frac{(k+1)^2}{(k+1)^3 + \sin(k+1)} x^{2(k+1)} \cdot \frac{(k^3 + \sin(k))}{k^2} x^{2k} = x^2 \lim_{k \rightarrow \infty} \frac{(k+1)^2}{k^2} \frac{k^3 + \sin(k)}{(k+1)^3 + \sin(k+1)}$$

This limit = 1

Converges for $|x| < 1$

AT Boundary $x = \pm 1$

Apply limit comparison test
 $\sum_{k=1}^{\infty} \frac{k^2}{k^3 + \sin(k)}$ $a_k = \frac{k^2}{k^3 + \sin(k)}$ $b_k = \frac{1}{k}$ both Diverge

(iii)

Apply Root TEST

$$\lim_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} = \lim_{k \rightarrow \infty} |e^{-k \ln k}|^{\frac{1}{k}} = \lim_{k \rightarrow \infty} |e^{-\ln k}|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{1}{k} \right| = 0$$

Converges for all x

Problem 3:

Consider the following parametric curve:

$$(x(t), y(t)) = (3 \cos(2t) - 2 \sin(3t), 3 \sin(2t) + 2 \cos(3t))$$

which looks like the following:

(i) Compute the tangent (velocity) vector to the curve.

$$(x', y') = (-6 \sin 2t - 6 \cos 3t, 6 \cos 2t + 6 \sin 3t)$$

(ii) Compute the length of the velocity vector **HINT:** The identity $\cos(a) \sin(b) - \sin(a) \cos(b) = \sin(b - a)$ may be useful.

$$\begin{aligned} (x')^2 + (y')^2 &= 36 \sin^2 2t + 72 \sin 2t \cos 3t + 36 \cos^2 3t \\ &\quad + 36 \cos^2 2t - 72 \cos 2t \sin 3t + 36 \sin^2 3t \end{aligned}$$

$$= 36 + 36 + 72 (\sin 2t \cos 3t - \cos 2t \sin 3t)$$

$$= 72 (1 + \sin(-t)) = 72 (1 - \sin t)$$

(iii) Where does the tangent vector have zero length?

$$\text{WHEN } \sin t = 1 \Rightarrow t = 0, 2\pi, 4\pi, \dots$$

(iv) Compute the integral

$$\int_0^{2\pi} y \frac{dx}{dt} dt.$$

What area does this represent?

$$\int_0^{2\pi} (3 \sin 2t + 2 \cos 3t) (-6 \sin 2t - 6 \cos 3t) dt$$

$$= \pi(-18 - 12) = \boxed{-30\pi}$$

(v) Compute the arclength of the curve over $t \in (0, 2\pi)$

$$\int_0^{2\pi} \sqrt{(x')^2 + (y')^2} dt = \int_0^{2\pi} \sqrt{72(1 - \sin t)} dt$$

$$= \int_0^{2\pi} \sqrt{72} \sqrt{1 - \sin t} dt$$

$$= \int_0^{2\pi} \sqrt{72} \sqrt{\frac{(1 - \sin t)(1 + \sin t)}{1 + \sin t}} dt$$

$$= \int_0^{2\pi} \sqrt{72} \frac{\sqrt{1 - \sin^2 t}}{\sqrt{1 + \sin t}} dt = \sqrt{72} \int_0^{2\pi} \frac{|\cos t|}{\sqrt{1 + \sin t}} dt$$

OR: $\sin(t + \frac{\pi}{2}) = \cos t$

$$= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{72} \sqrt{1 - \sin(t + \frac{\pi}{2})} dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{72} \sqrt{1 - \cos(t + \frac{\pi}{2})} dt$$

use $1 - \cos t = 2 \sin^2 \frac{t}{2}$

$$= \sqrt{72} \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1 + \sin t}} dt$$

$$= 4\sqrt{72} \sqrt{1 + \sin t} \Big|_0^{\frac{\pi}{2}}$$

$$= 4\sqrt{72} \sqrt{2} = \boxed{48}$$

Problem 4:

Evaluate the convergence or divergence of the following series

(i)

$$\sum_{k=2}^{\infty} \frac{1}{k + k^{1/2} \sin(k)}$$

Limit comparison Test $\sum \frac{1}{k}$
Divergent.

(ii)

~~$$\sum_{k=0}^{\infty} \frac{1}{k + k^{3/2} \sin(k)}$$~~
HARD

Typo. Supposed to be

$$\sum \frac{\sin k}{k + k^{3/2}}$$

(iii)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k + k^{1/2}}$$

Converges by alt. Series Test

$$a_k = \frac{1}{k + k^{1/2}} \text{ is}$$

• Decreasing ✓ (since $k, k^{1/2}$ are increasing)

• $a_k > 0$ ✓

• $\lim_{k \rightarrow \infty} a_k = 0$.

(iii)

Problem 5:

Evaluate the following limits using Taylor series: (i)

$$\lim_{x \rightarrow 0} \frac{e^{x^2} - 1 - x^2}{x^2} = 0 \quad e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \dots$$

$$e^{x^2} - 1 - x^2 = \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

$$\frac{e^{x^2} - 1 - x^2}{x^2} = \frac{x^2}{2!} + \dots$$

(ii)

$$\lim_{x \rightarrow 0} \frac{\sin(x^2) - x^2}{x^6}$$

(iii)

$$\lim_{x \rightarrow 0} \frac{\cos(2x) - 1 - 2x^2}{x^6}$$

$$\frac{\sin x^2 - x^2}{x^6} = \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \dots - x^2}{x^6}$$

$$= -\frac{1}{3!} + \frac{x^4}{5!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{\sin x^2 - x^2}{x^6} = -\frac{1}{3!}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots$$

$$\cos 2x - 1 - 2x^2 = -4x^2 + \frac{16x^4}{4!} - \dots$$

$$\frac{\cos 2x - 1 - 2x^2}{x^6} = \frac{-4}{x^4} + \dots \quad \infty$$

Problem 6:

Find the Taylor series for the following functions about the indicated point

(i) $f(x) = \sin(x)$ $c = \frac{\pi}{4}$

$$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'\left(\frac{\pi}{4}\right) = +\cos\frac{\pi}{4} = +\frac{\sqrt{2}}{2}$$

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\vdots$$

$$f = \frac{\sqrt{2}}{2} \left(1 + (x - \frac{\pi}{4}) - \frac{(x - \frac{\pi}{4})^2}{2!} + \frac{(x - \frac{\pi}{4})^3}{3!} + \frac{(x - \frac{\pi}{4})^4}{4!} + \frac{(x - \frac{\pi}{4})^5}{5!} + \dots \right)$$

Pattern of signs $+ + - + - \dots$

(ii) $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ $c = 0$ **HINT:** Recall that $\ln(a/b) = \ln(a) - \ln(b)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$\ln(1+x) - \ln(1-x) = \ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \dots = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

(iii) ~~$f(x) = \ln\left(\frac{1+x}{1-x}\right)$~~ $c = 0$ **Hint:** This is tricky! What is $\frac{d^4 f}{dx^4}$

Same as (ii)

Problem 7:

Consider the following parametric curve:

$$(x(t), y(t)) = (t - \sin(3t), 1 - \cos(3t))$$

which looks like the following:

(i) Compute the tangent (velocity) vector to the curve.

$$(x', y') = (1 - 3 \cos 3t, \sin 3t)$$

(ii) Compute the length of the velocity vector **HINT:** The identity $\cos(a) \sin(b) - \sin(a) \cos(b) = \sin(b - a)$ may be useful.

$$(x')^2 + (y')^2 = (1 - 3 \cos 3t)^2 + \sin^2 3t$$

$$1 + 9 \cos^2 3t + \sin^2 3t$$

Never! $v' = 0 \Rightarrow \sin 3t = 0 \Rightarrow \cos 3t = \pm 1$
but the $x' \neq 0$.

(iii) Where does the tangent vector have zero length?

(iv) Compute the integral

$$\int_0^{2\pi} y \frac{dx}{dt} dt.$$

What area does this represent?

$$y = 1 - 3\cos 3t$$
$$\frac{dx}{dt} = 1 - 3\cos 3t$$

$$\int y \frac{dx}{dt} = \int (1 - 3\cos 3t)^2$$
$$= \int_0^{2\pi} 1 - 6\cos 3t + 9\cos^2 3t$$
$$= 2\pi + 9\pi = 11\pi$$

(v) ~~Compute the arclength of the curve over $t \in (0, 2\pi)$~~

This is NOT doable
w/ the Typo.