Math 285 Homework # 3
Due Friday Sept. 5 in class.

Section 1.4 # 3,14,17,36

Section 1.5 # 4,8,15,16,23,42

Section 1.6 # 16,35,59

1.4.3

$$\frac{dy}{dx} = y\sin(x)$$
$$\int \frac{dy}{y} = \int \sin(x)dx$$
$$\ln|y| = -\cos(x) + c$$
$$y = Ae^{-\cos(x)}$$

1.4.14

$$\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$$
$$\int (1+\sqrt{y})dy = \int (1+\sqrt{x})dx$$
$$y + \frac{2}{3}y^{\frac{3}{2}} = x + \frac{2}{3}x^{\frac{3}{2}} + c$$

1.4.17

$$\frac{dy}{dx} = 1 + x + y + xy$$
$$= (1+x)(1+y)dx$$
$$\int \frac{dy}{1+y} = \int (1+x)dx$$
$$\ln|1+y| = x + \frac{x^2}{2} + c$$
$$y = c\exp(x + \frac{x^2}{2}) - 1$$

1.4.36

$$N(t) = N(0)\exp(-kt)$$

From the text $N(0) = 5 \times 10^{10}$, $N(t) = 4.6 \times 10^{10}$, $k = .0001216$. Solving for t gives

$$t = \frac{\ln(4.6/5.0)}{.0001216} \approx 686$$

Your conclusions here.....

1.5.4

$$\begin{aligned} y' - 2xy &= \exp(x^2) \\ y' \exp(-x^2) - 2xy \exp(-x^2) &= 1 \\ \frac{d}{dx}(y \exp(-x^2)) &= 1 \\ (y \exp(-x^2) &= x + c \\ y &= x \exp(x^2) + c \exp(x^2) \end{aligned}$$

1.5.8

$$\begin{aligned} 3xy' + y &= 12x \\ y' + \frac{1}{3x} y &= 4 \\ y' x^{\frac{1}{3}} + \frac{1}{3} y x^{-\frac{2}{3}} = 4 \frac{d}{dx}(y x^{\frac{1}{3}}) &= 4 \\ (y x^{\frac{1}{3}}) &= 4x + c \\ y &= 4x^{\frac{2}{3}} + c x^{-\frac{1}{3}} \end{aligned}$$

1.5.16

$$\begin{aligned} y' &= (1 - y)\cos(x) \ y(\pi) = 2 \\ -\ln|1 - y| &= \sin(x) + c \ c = -ln|1| = 0 \\ y = 1 + \exp(-cos(x)) \end{aligned}$$

1.5.23

$$\begin{aligned} xy' + (2x - 3)y &= 4x^4 \\ y' + (2 - \frac{3}{x})y &= 4x^3 \end{aligned}$$

The integrating factor is given by

$$\mu(x) = e^{\int (2 - \frac{3}{x}) dx} = e^{2x - 3\ln(x)} = \frac{e^{2x}}{x^3}$$

this gives

$$\begin{aligned}
\frac{e^{2x}}{x^3}y' + \frac{e^{2x}}{x^3}(2 - \frac{3}{x})y &= 4x^3\frac{e^{2x}}{x^3} \\
\frac{d}{dx}\left(y\frac{e^{2x}}{x^3}\right) &= 4e^{2x} \\
y\frac{e^{2x}}{x^3} &= 2e^{2x} + c \\
y &= 2x^3 + cx^3e^{-2x}
\end{aligned}$$

1.5.42 Using the fact that the volume is $v = \frac{4}{3}\pi r^3$ one finds that the mass of the hailstone is given by

$$m(t) = \frac{4}{3}\pi k^3 t^3 \rho$$

Plugging this into Newton's law in the form

$$\frac{d}{dt}\left(m(t)v\right) = m(t)g$$

gives

$$\frac{d}{dt}\left(t^3 v\right) = t^3 g$$

Integrating this up and plugging in $v(0) = 0$ gives

$$t^3 v = \frac{t^4}{4}g + c = \frac{t^4}{4}g$$

knowing that a particle falling under gravity usually satisfies $v = gt$ we can conclude that the hailstone accelerates as if gravity were $\frac{1}{4}$ as strong.

1.6.16

$$y' = \sqrt{x + y + 1}$$

Making the substitution $w^2 = x + y + 1$ and thus $2ww' = 1 + y'$ gives

$$\begin{aligned}
2ww' - 1 &= w \\
\int \frac{2wdw}{1 + w} &= dx \\
2w - 2\ln(1 + w) &= x + c \\
2\sqrt{x + y + 1} - 2\ln(1 + \sqrt{x + y + 1}) &= x + c
\end{aligned}$$

1.6.35

$$\underbrace{(x^3 + \frac{y}{x})}_{Q(y,x)}\, dx + \underbrace{(y^2 + \ln(x))}_{P(y,x)}\, dy$$

We'd like to check if this is exact: if there is $F(x,y)$ such that $\frac{\partial F}{\partial x} = Q(x,y)$ and $\frac{\partial F}{\partial y} = P(x,y)$. A necessary and sufficient condition is the equality of the mixed partials. Computing we find that

$$\frac{\partial P}{\partial x} = \frac{1}{x} = \frac{\partial Q}{\partial y}$$

and thus there is such a function F. In order to find it we integrate one of the equations

$$\begin{aligned}\frac{\partial F}{\partial y} &= (y^2 + \ln(x)) \\ F &= \frac{y^3}{3} + y\ln(x) + C(x) \\ \frac{\partial F}{\partial x} &= \frac{y}{x} + C'(x) = x^3 + \frac{y}{x} \\ C(x) &= \frac{x^4}{4}\end{aligned}$$

Thus the solution is given by

$$F(x,y) = \frac{y^3}{3} + y\ln(x) + \frac{x^4}{4} = \text{constant}$$

which defines y implicitly as a function of x. One cannot explicitly solve for y as a function of x.

1.6.59 The easiest way to solve

$$\frac{dy}{dx} = \frac{x - y - 1}{x + y + 3}$$

is to multiply through by $x + y + 3$ to get

$$\underbrace{(x + y + 3)}_{P(x,y)}\, dy + \underbrace{(1 + y - x)}_{Q(x,y)}\, dx = 0$$

It is easy to see that this is exact:

$$\frac{\partial Q}{\partial y} = 1 = \frac{\partial P}{\partial x}$$

The we solve

$$\begin{aligned}
\frac{\partial F}{\partial y} &= (x+y+3) \\
F &= xy + \frac{y^2}{2} + 3y + C(x) \\
\frac{\partial F}{\partial x} &= y + C'(x) = 1 + y - x \\
C'(x) &= 1 - x \\
C(x) = x - \frac{x^2}{2}
\end{aligned}$$

Thus the solution is

$$F(x,y) = xy + \frac{y^2}{2} + 3y + x - \frac{x^2}{2} = \text{constant}$$