

Math 285 Lecture 1: Ordinary Differential Equations

An ordinary differential equation is any relationship between a function (usually denoted $y(x)$) and its derivatives up to some order:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^k y}{dx^k}\right) = 0$$

For example the following are all ordinary differential equations.

$$\frac{dy}{dx} = -ky \tag{1}$$

$$\frac{d^2y}{dx^2} = -y \tag{2}$$

$$\frac{d^5y}{dx^5} = -y^3 \tag{3}$$

$$\frac{d^8y}{dx^8} + y \frac{dy}{dx} = y \tag{4}$$

$$y^2 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^{11}y}{dx^{11}}\right)^2 \tag{5}$$

Some basic terminology:

Order: The order of a differential equation is the order of the highest derivative which appears in the equation. The order of the equations above are 1, 2, 5, 8 and 11 respectively.

Linear vs. Nonlinear: An equation is linear if the dependent variable and all its derivatives enter into the equation *linearly*. The dependence on the independent variable can be arbitrary. In the examples above the first two are linear, since y, y', y'' all enter linearly. The third equation is nonlinear because there is a y^3 term (which is nonlinear). The fourth equation is nonlinear because of the $y \frac{dy}{dx}$ term.

A linear equation can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots a_0(x) y = 0.$$

The related equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots a_0(x) y = f(x).$$

is technically not linear ($f(x)$ is not linear in y or its derivatives) but is covered by the same basic theory: it is called a linear inhomogeneous equation.

A great deal of time in this class will be spend learning to solve linear differential equations. Solving nonlinear differential equations is in general quite difficult, although certain special kinds can be solved exactly.

Differential equations arise in many physical contexts where the rate of change of a quantity can be related to the quantity itself. For example:

Example 1. Newton's Law of Cooling

Newton's law of cooling states that the rate of change of the temperature of a body is proportional to the difference in temperature between the body and

the surrounding medium. What is the differential equation which governs the temperature of the body?

Let T denote the temperature of the body. The rate of change of the temperature is obviously $\frac{dT}{dt}$. If the temperature of the surrounding medium is denoted by T_0 then we have the equation

$$\frac{dT}{dt} = -k(T - T_0)$$

where $-k$ is the constant of proportionality. We put the $-$ sign in there because it is clear that the temperature of the body should converge to that of the medium: If $T > T_0$ then the body should be getting cooler (the rate of change should be negative)

Another example comes from freshman physics:

Example 2. Newton's Law of Motion

(You can't get away from Newton in this class)

Newton's laws of motion state that

- Force = Mass \times acceleration $F = ma$.
- Force is minus the derivative of the potential energy

So we have

$$ma = F = -\frac{dV(y)}{dy} \tag{6}$$

$$a = \frac{d^2y}{dt^2} \tag{7}$$

giving

$$\frac{d^2y}{dt^2} = -\frac{dV(y)}{dy}$$

This is, of course, a differential equation for the position y . It is second order and nonlinear (unless, of course $V(y)$ is a quadratic function of y , in which case the equation is linear. This is called the simple harmonic oscillator.

Important thing to Note: Verification

While it can be very difficult to solve a general ordinary differential equation it is usually pretty easy to check whether or not a given function $y = f(x)$ solves the equation, since one can simply compute all derivatives and check whether or not the equation is satisfied.

Example 3. Check that the function

$$y(x) = A \cos(x) + B \sin(x)$$

satisfies the equation

$$\frac{d^2y}{dx^2} = -y$$

Similarly we have another example

Example 4. Check that the function

$$y(x) = \frac{1}{\cos(x - x_0)}$$

satisfies the equation

$$\frac{d^2y}{dx^2} = 2y^3 - y$$

A General Principle: ORDER = NUMBER OF FREE PARAMETERS

In general we expect the general solution to a differential equation to involve some constants of integration. If the equation is of n^{th} order we typically expect the solution to have n constants of integration. This makes a certain amount

of sense - if we want to recover the function given the n^{th} derivative we need to integrate up n times, which introduces n constants of integration. Another way to think about this is the following: **(Solve for successive derivatives.)**

(This is always true for linear equations, and also true for nonlinear equations with “well-behaved” nonlinearities.

Thus is the first example above we have two arbitrary constants (A, B) so we expect that this is the most general solution. In the second example we have only one constant of integration x_0 , so we can guess that this is *not* the most general solution. (In fact the most general solution is something called a Jacobi elliptic function).

Practice Exercises:

- Write down a differential equation which is different from the above examples. Give the order of the equation, and state whether it is linear or nonlinear.
- Check that the function $y(x) = 1 - x^2$ satisfies the differential equation $y - xy' = 1 + x^2$
- Suppose that $y(x)$ satisfies $y'' = y$ together with the initial conditions $y(0) = 1, y'(0) = 0$. What is $\frac{d^k y}{dx^k}(0)$ as a function of k ?