

Lecture 11: Math 285 (Bronski)

Independence and the Wronskian:

The next thing to notice is that the Wronskian actually satisfies a first order equation, so we can explicitly write down the solution:

Theorem: Suppose that y_1 and y_2 are two solutions to the equation

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Then the Wronskian $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ solves the differential equation

$$A(x)W' + B(x)W = 0$$

Proof

Example 1. Since $\sin(x)$ and $\cos(x)$ are both solutions of

$$y'' + y = 0$$

- in other words $A(x) = 1$ and $B(x) = 0$ it follows that the Wronskian solves

$$W' = 0$$

in other words the Wronskian is a constant independent of x . We already know this to be the case.

Example 2. We saw last time that two solutions to

$$x^2y'' - 4xy' + 6y = 0$$

are $y_1 = x^2$ and $y_2 = x^3$. It follows that the Wronskian solves

$$x^2W' - 4xW = 0$$

which has the solution $W = cx^4$. Computing the Wronskian of y_1 and y_2 gives

Notice that, since the Wronskian is given by

$$W(x) = W(0)e^{-\int p(x)dx}$$

we have the following result:

Theorem: Suppose that y_1 and y_2 are two solutions to the equation

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous in some interval I . Then the Wronskian $W(x)$ is either identically 0 in I or it is **NEVER** zero in I .

proof: The Wronskian satisfies the equation

$$W' + p(x)W = 0$$

This is a first order equation and can be integrated up to give

$$W(x) = W(0)e^{\int p(x)dx}$$

Since $p(x)$ is a continuous function the integral is bounded and thus the integrand is some finite number, and the exponential never vanishes. Thus the only way that $W(x)$ can vanish is if $W(0)$ vanishes, in which case the Wronskian is identically zero.

The reason for studying the Wronskian is the following: linear independence and non-vanishing of the Wronskian are closely connected with the uniqueness of solutions of linear differential equations. This is the content of the next theorem:

Theorem 1. Consider the initial value problem

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous in an interval I and suppose that y_1 and y_2 are two linearly independent solutions. Then **ANY** solution of the above can be written as a linear combination of y_1, y_2 .

Let's do a few examples:

Example 3. *The following are all pairs of solutions to the equation*

$$4y'' = 9y$$

$$y_1 = e^{3x/2} \quad y_2 = e^{-3x/2} \quad (1)$$

$$y_1 = \cosh(3x/2) \quad y_2 = \sinh(3x/2) \quad (2)$$

$$y_1 = e^{-3x/2} \quad y_2 = 5e^{-3x/2} \quad (3)$$

In the first two cases the Wronskian is non-zero, so any solution can be written as a linear combination of $e^{3x/2}$ and $e^{-3x/2}$ or of $\cosh(3x/2)$ and $\sinh(3x/2)$.

Constant Coefficient Second Order

The simplest second order linear ordinary differential equation is one where the coefficients are simply constant: that is

$$ay'' + by' + cy = 0 \quad (a \neq 0).$$

These equations are nice because we can always just write down the general solution. We introduce something called the characteristics equation or the characteristic polynomial of the problem.

Definition 1. *The characteristic equation of the equation*

$$ay'' + by' + cy = 0$$

is given by

$$ar^2 + br + c = 0$$

The motivation for this is follows: We can always try to “guess” a solution $y = e^{rx}$. Plugging this in gives

$$y = e^{rx} \tag{4}$$

$$y' = re^{rx} \tag{5}$$

$$y'' = r^2 e^{rx} \tag{6}$$

and thus this gives a solution if and on if r satisfies the equation

$$ar^2 + br + c = 0.$$

There are three different cases to consider: they are

- Two real roots.
- One (real) double root.
- Complex conjugate pair of roots.

The first case is the easiest, so we'll consider it first.