Lecture 12: Math 285 (Bronski)
Constant Coefficient Second Order

The simplest second order linear ordinary differential equation is one where the coefficients are simply constant: that is

$$ay'' + by' + cy = 0 \qquad (a \neq 0).$$

These equations are nice because we can always just write down the general solution. We introduce something called the characteristics equation or the characteristic polynomial of the problem.

Definition 1. *The characteristic equation of the equation*

$$ay'' + by' + cy = 0$$

is given by

$$ar^2 + br + c = 0$$

The motivation for this is follows: We can always try to "guess" a solution $y = e^{rx}$. Plugging this in gives

$$y = e^{rx} \tag{1}$$
$$y' = re^{rx} \tag{2}$$
$$y'' = r^2e^{rx} \tag{3}$$

and thus this gives a solution if and on if r satisfies the equation

$$ar^2 + br + c = 0.$$

There are three different cases to consider: they are

- Two real roots.
- One (real) double root.
- Complex conjugate pair of roots.

The first case is the easiest, so we'll consider it first.

In the first case, suppose that the two roots are r_1, r_2. Then we can compute the Wronskian

which never vanishes. This implies that the two solutions $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent, and thus the general solution is given by

$$y = ae^{r_1 x} + be^{r_2 x}$$

Example 1. *A mass-spring-dashpot system has the following dynamics: there is a force* $-kx$ *due to a spring and a resisting force* $-rv = -rx'$ *due to a dashpot. Thus the system is governed by*

$$m\frac{d^2x}{dt^2} = -kx - r\frac{dx}{dt}$$

Suppose that $m = 1, r = 3, k = 2$, *and that the initial displacement is* $x(0) = 1$ *and the initial velocity* $\frac{dx}{dt}(0) = 0$. *Find the displacement as a function of time.*

How long is it until the displacement is $\frac{1}{10}$?

Repeated Roots:

Consider the following differential equation:

$$y'' + 2y' + y = 0$$

The characteristic equation is given by

$$r^2 + 2r + 1 = 0$$

which has $r = -1$ as a double root. This gives us one solution, but not a second solution. So how do we find the second solution?

I'll give the easiest method which is a "factoring" trick. We'll define $w = y' + y$. Then note that

$$w' + w = y'' + y' + y' + y = 0$$

Thus w satisfies $w' + w = 0$. This is a first order equation. We can integrate it up to find the solution $w = ae^{-x}$. But $w = y' + y$. So we have the equation

$$y' + y = ae^{-x}$$

This has the integrating factor e^x. Multiplying through by the integrating factor gives

$$y'e^x + ye^x = a \tag{4}$$
$$(ye^x)' = a \tag{5}$$
$$ye^x = ax + b \tag{6}$$
$$y = (ax + b)e^{-x} \tag{7}$$

Well, we next need to check that these are really independent solutions, so we should compute the Wronskian.

This leads to the next theorem:

Theorem 1. *Suppose that the constant coefficient differential equation*

$$ay'' + by' + cy = 0$$

has a characteristic equation with a single root r_1 *of multiplicity* 2*. (in other words* $b^2 - 4ac = 0$ *Then the two linearly independent solutions to the above are*

$$e^{r_1 x} \qquad xe^{r_1 x}$$