

Lecture 2: Math 285 (Bronski)

Last time we began discussing differential equations. Recall the the order of a differential equation is the order of the highest derivative. The general rule of thumb is that the most general solution to a n^{th} order differential equation involves n constants of integration. While there are some exceptions to this for the vast majority of differential equations you will encounter in practice this is true. (We will state a theorem later which says essentially this). Usually we will impose n initial conditions - we will choose the value of the function at some point as well as $n - 1$ derivatives.

There are basically really only two techniques for solving differential equations. The first is to recognize the equation as representing the exact derivative of something, and integrating it up. The second is to guess. In this class we'll learn lots of tricks which will help to accomplish one or the other of these techniques.

Today we are mostly going to focus on the first technique: recognizing the equation as representing a exact derivative. There are a number of types of equation that often arise in practice where this is possible.

The simplest example is when one has some derivative of y equal to a function of x . This occasionally arises in applications. For such equations we can simply integrate up a number of times.

Example 1. *Suppose that the height $y(t)$ of a falling body evolves according to*

$$\frac{d^2y}{dt^2} = -g$$

Find the height as a function of time.

Integrating up once we find that

$$\frac{dy}{dt} = -gt + v$$

where v is a constant of integration (representing the velocity of the body at time $t = 0$: $v = y'(0)$). Integrating up a second time gives the equation

$$y = -\frac{gt^2}{2} + vt + h$$

where $h = y(0)$ is a second constant of integration representing the initial height of the body.

More usually a differential equation will involve both the independent and the dependent variable. It still sometimes happens that one can rewrite the equation so that it can be recognized as the derivative of something.

Example 2. *Solve the equation*

$$\frac{dT}{dt} = -k(T - T_0)$$

arising in Newton's law of cooling. Discuss the form the solution takes

A similar equation in which the solution depends on both the independent and the dependent variables is the following:

Example 3. Solve the equation

$$\frac{dy}{dx} = xy \quad y(0) = 1$$

we can rewrite this as

$$\frac{dy}{y} = x$$

Integrating from $x = 0$ up to x gives

$$\int \frac{dy}{y} = \int x dx \tag{1}$$

$$\ln(y(x)) = \frac{x^2}{2} + C \tag{2}$$

So

$$y(x) = y(0)e^{\frac{x^2}{2}} = e^{\frac{x^2}{2}}$$

The most general first order equation which can be solved this way is called a separable equation, and takes the form

$$\frac{dy}{dx} = f(y)g(x)$$

One can run through the same procedure and find that

$$\int \frac{dy}{f(y)} = \int g(x)dx + C$$

Then one can attempt to solve for y as a function of x . This may or may not be possible, but typically it will at least define y as a function of x locally by the implicit function theorem.

Example 4. Solve the equation

$$\frac{dy}{dx} = (y^2 + y)x \quad y(2) = 1$$

We can solve this by writing it in the form

$$\frac{dy}{y^2 + y} = x dx$$

and integrating up to get

$$\int_{y(2)}^{y(x)} \frac{dy}{y^2 + y} = \int_2^x x dx \quad (3)$$

$$\int_{y(2)}^{y(x)} \frac{1}{y} - \frac{1}{y+1} dy = \int_2^x x dx \quad (4)$$

$$\ln(y) - \ln(y+1) - \ln(1) + \ln(2) = \frac{x^2}{2} \Big|_2^x \quad (5)$$

Exponentiating the above gives

$$\frac{y}{y+1} = e^{\frac{x^2}{2} - 2 - \ln(2)} = 2e^{\frac{x^2}{2} - 2}$$

This can be solved for y to give

$$y = \frac{2e^{\frac{x^2}{2} - 2}}{1 - 2e^{\frac{x^2}{2} - 2}}$$

Another example where it is not really possible to solve for y explicitly as a function of x is given by the following:

Example 5. Solve the equation

$$\frac{dy}{dx} = \frac{x^2 + 5}{y^3 + 2y} \quad y(0) = 1$$

One has to be careful in applying this idea. Usually to put the equation in the form of an exact derivative one must divide through by some quantity. As you are probably familiar with from algebra, if one multiplies through by something you must be careful that you are not introducing extraneous roots, or removing relevant roots. If one is not careful sometimes one can miss solutions by making such implicit assumptions. Here is an example

Example 6. *Solve the differential equation*

$$\frac{dy}{dx} = y^{\frac{1}{3}} \quad y(0) = 0$$

Since this is a separable equation we can write it as

$$\frac{dy}{y^{\frac{1}{3}}} = dx$$

Which integrates up to

$$\frac{3}{2}y^{\frac{2}{3}} = x + c$$

when $x = 0$ we have $y = 0$, giving $c = 0$. Thus we have

$$y = \left(\frac{2x}{3}\right)^{\frac{3}{2}}$$

There is only one problem: This is NOT the only solution - we missed the solution $y(x) = 0$.

*This should be a little bit unsettling. If a differential equation is modeling a physical system, such as a trajectory, there should be a **unique** solution: physically there is only one trajectory. Here we have a differential equation with more than one solution. So what happened?*

Well, when we divide through by $y^{\frac{1}{3}}$ we are making the implicit assumption that this quantity is not identically zero. So we need to go back and check that $y = 0$ is not a solution to the equation. In this case it is a solution to the equation.

Later we will state a fundamental existence and uniqueness theorem. This will show that for “nice” differential equations this simply does not happen: given an initial condition there is one and only one solution.

Practice Exercises:

- Solve the differential equation $y' = xy \ln |y|$
- Suppose that a projectile is fired upwards at 100 meters per second, and that the acceleration due to gravity is 10 meters per second ². At what time does the projectile hit the ground.
- If the projectile has a horizontal velocity of 20 meters per second how far has it traveled when it strikes the ground.