

Lecture 5: Math 285 (Bronski)

More on First Order Linear Equations

Last time we considered the first order linear inhomogeneous equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This can be solved by multiplying through by an “integrating factor” and realizing the result as an exact derivative. In the above case the integrating factor is

$$e^{\int P(x)dx}$$

There is an easy way to remember this: it is the reciprocal of the solution to the homogeneous problem $e^{-\int P(x)dx}$. Multiplying through by this integrating factor gives

$$\frac{d}{dx} e^{\int P(x)dx} y + P(x)y e^{\int P(x)dx} = Q(x)e^{\int P(x)dx} \quad (1)$$

$$\frac{d}{dx} (y e^{\int P(x)dx}) = Q(x)e^{\int P(x)dx} \quad (2)$$

$$y = \underbrace{e^{-\int P(x)dx} \int Q(x)e^{\int P(x)dx} dx}_{\text{particular}} + \underbrace{c e^{-\int P(x)dx}}_{\text{homogeneous}} \quad (3)$$

Example 1. Example: Mixture Problem

A tank contains 10 kilograms of salt dissolved in 100 liters of water. Water containing .001 kilogram of salt per liter flows into the tank at 10 liters per minute, and well mixed water flows out at 10 liters per minute. What is the differential equation governing the amount of salt in the tank at time t ? Solve it

Let $Y(t)$ represent the total amount of salt (in kilograms) in the water. The total amount of water in the tank is a constant 100 liters. The amount of salt flowing into the tank is .01 kg/min. The amount flowing out depends on the concentration of the salt. If the amount of salt is $Y(t)$ then the concentration is $Y(t)/100$, and the amount flowing out is $10Y(t)/100 = Y(t)/10$. So the differential equation is

$$\frac{dY}{dt} = .01 - \frac{Y}{10}$$

The integrating factor is $e^{\int \frac{1}{10} dt} = e^{\frac{t}{10}}$ giving

$$\frac{d}{dt} (Y e^{\frac{t}{10}}) = .01 e^{\frac{t}{10}} \quad (4)$$

$$(Y e^{\frac{t}{10}}) = .1 e^{\frac{t}{10}} + c \quad (5)$$

$$Y = .1 + c e^{-\frac{t}{10}} \quad (6)$$

$$(7)$$

Initially we have $Y(0) = 10$ (initially there are 10 kg of salt) thus

$$Y = .1 + 9.9e^{-\frac{t}{10}}$$

So that the total amount of salt is asymptotic to .1kg. This is kind of obvious. (Why?)

A more interesting problem occurs when the water flows into the tank at a different rate than it flows out. For instance.

Example 2. Suppose now that the water flows out of the tank at 5 liters per minute. What is the equation governing the amount of salt in the tank? Solve it!

In this case the amount of water in the tank is now $100 + 5t$, and is not constant. In this case the differential equation becomes:

Exact Equations (Conservative vector fields)

I want to mention one more type of equation that comes up, called exact equations. These are equations of a special type: equations of the form

$$M(x, y) \frac{dy}{dx} + N(x, y) dx = 0$$

where M, N are related in the following way: There is some function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial y} = M(x, y) \quad (8)$$

$$\frac{\partial \psi}{\partial x} = N(x, y) \quad (9)$$

These kinds of equations are connected with vector calculus and the problem of determining if a vector field is a gradient (if you have had any exposure to this is mathematics or (more likely) physics).

First question: How can I recognize such an equation? How do I know if there is such a function? Well, I can use vector calculus! I know about the equality of mixed partials. Differentiating the above equations with respect to x and y respectively gives

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial M}{\partial x}(x, y) \quad (10)$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial N}{\partial y}(x, y) \quad (11)$$

So a necessary condition is that $M_x = N_y$. It turns out that this is also a sufficient condition.¹ If we can find such a function ψ then by the chain rule the above equation is equivalent to

$$\psi_y(x, y) \frac{dy}{dx} + \psi_x(x, y) dx = 0 \quad (12)$$

$$d(\psi(x, y)) = 0 \quad (13)$$

$$\psi(x, y) = c \quad (14)$$

This makes geometric sense - the above equation is equivalent to

$$\nabla \psi \cdot (dx, dy) = 0$$

so the directional derivative of ψ is constant. This is equivalent to saying that ψ is constant, or that (x, y) are level sets of ψ .

So how do we find ψ ? Well, the inverse to partial differentiation is partial integration. This is easiest to illustrate with an example

Example 3. Solve the equation

$$(y^3 + 3x^2y)dy + (3y^2x)dx = 0$$

¹On simply connected domains, for those of you in the know....

We'd like to find a function $\psi(x, y)$ such that

$$\frac{\partial\psi}{\partial y} = y^3 + 3x^2y \quad (15)$$

$$\frac{\partial\psi}{\partial x} = 3y^2x \quad (16)$$

Well, first we have to check that this is possible. The necessary condition is that the mixed partials ought to be equal. Differentiating gives

So that works. Thus we know that there is such a ψ . Now we have to find it. We can start by integrating up

$$\psi = \int y^3 + 3x^2y dy = \frac{y^4}{4} + \frac{3x^2y^2}{2} + f(x)$$

We still don't know $f(x)$, but we can use the other equation

$$\psi_x = 3xy^2 + f'(x)$$

comparing this to the above gives $f'(x) = 0$ so $f(x) = c$. Thus the solution is given by

$$\psi(x, y) = \frac{y^4}{4} + \frac{3x^2y^2}{2} = c$$

Note that this equation is IMPLICIT. You can actually solve this for y as a function of x (The above is quadratic in y) but you can also leave it in this form.

Another example is the following:

Example 4.

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + y^2}$$

This is similar to an example we looked at last class, as an illustration of the existence/uniqueness theorem. We can rewrite this as

$$(x^2 + y^2)dy + (2xy)dx = 0$$

This is exact, since we can compute

Thus the solution is

$$\frac{y^3}{3} + x^2y = c$$

There are a number of other types of equations that can be integrated by various tricks. One class of equations is those of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

These fall to the substitution $v = \frac{y}{x}$ or $y = xv$. This leads to

$$\frac{dy}{dx} = x \frac{dv}{dx} + v = F(v)$$

which is a separable equation.

An example of this type is

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

Making the substitution $y = xv$ gives

$$x \frac{dv}{dx} + v = \frac{x^2v}{x^2 + x^2v} = \frac{v}{1 + v^2}$$

Which is separable.