Lecture 6: Math 285 (Bronski)

section 1.6 Exact equations and substitution methods

There are a lot of substitution methods outlined in this section. Hardly any of them are worth memorizing. The basic idea is always the same: look at the form of the equation. See if it suggests anything. Don't be afraid to play around with the equation and see if you can make it simpler.

Some examples:

Example 1. *Consider the equation*

$$\frac{dy}{dx} = -\frac{x^2 + 2xy + y^2}{1 + (x+y)^2}$$

Well, it is clear that the righthand side is only a function of the sum $x+y$. This suggests making the substitution $v = y + x$. The equation above becomes

$$\frac{dv}{dx} - 1 = -\frac{v^2}{1+v^2} \tag{1}$$

$$\frac{dv}{dx} = \frac{1}{1+v^2} \tag{2}$$

$$(1+v^2)dv = dx \tag{3}$$

Integrating this up gives the equation

$$v + \frac{v^3}{3} = x + c \tag{4}$$

$$\frac{y}{x} + \frac{1}{3}\frac{y^3}{x^3} = x + c \tag{5}$$

Another example occurs for equations of second order when either the dependent variable y or the independent variable x is missing.

Example 2. *Consider the problem*

$$y'' + y'/x = x^5$$

This is a second order equation, but there is no y term. Thus we can make the substitution $v = y'$. This gives

$$v' + v/x = x^5$$

This is a first order linear (inhomogeneous) equation, and thus can be solved by multiplying through by an appropriate integrating factor. In this case the integrating factor is $e^{\int dx/x} = e^{\ln(x)} = x$ Thus

$$\frac{d}{dx}(xv) = x^6$$

Another similar example occurs when the independent variable x is missing.

$$F(y, y', y'') = 0$$

This can be simplified by the substitution

$$y' = p(y)$$

(in other words, instead of thinking of x as the independent variable we think of y as the independent variable. Differentiating the above gives

$$y'' = y'\frac{dp}{dy} = p\frac{dp}{dy}$$

for example

Example 3. *Solve the equation*

$$\frac{y'}{'} y^2 = y'$$

Making the change of variables above gives

$$y^2 p\frac{dp}{dy} = p$$

either $p = 0$ *or dividing through by* p *gives*

$$\frac{dp}{dy} = \frac{1}{y^2}$$

or $p(y) = -\frac{1}{y} + c$ *but this is the same as*

$$\frac{dy}{dx} = c - \frac{1}{y} = \frac{cy - 1}{y} \tag{6}$$

$$\frac{y}{cy - 1} dy = dx \tag{7}$$

which can be integrated up to get