Lecture 7: Math 285 (Bronski)

The phase line, equilibria, and stability

We've talked a lot about first order equations. I'm going to talk some more about equations of the form

$$y' = f(y).$$

These equations are always solvable in principle, as they are separable and can be reduced to integration, but it is often useful to have a way to understand how they behave without actually solving the equation. This is the idea of section 2.2. Essentially by drawing the direction field one can always understand qualitatively how the solutions to such an equation behave. This is often more useful than actually having a complicated solution formula.

First we define equilibria: an equilibrium is any constant solution. If $y = \textit{constant}$ then $y' = 0$, hence an equilibrium is a solution to $f(y) = 0$.

This should remind you of the last couple of problems on IODE project number one. There you saw that if $f(2) = 0$ then the function $y(x) = 2$ was a solution.

Example 1. *In the equation*

$$y' = y^3 - 3y + 2$$

the only equilibrium is $y = 1$

Example 2. *In the equation*

$$y' = (y - 1)(y + 2)(y - 3)$$

the equilibria are $y = 1, y = -2, y = 3.$

Since the right-hand side $f(y)$ doesn't depend on the variable x we can collapse the slope field to a slope line. In the following way:

We refer to this kind of diagram as the phase line.

When we did the IODE project we saw that if $f(y)$ vanished at a point then the solution $y(x)$ had a horizontal asymptote at that point either as $x \to \infty$ or as $x \to -\infty$. In the first case the function $y(x)$ approaches this value, in the second case the function $y(x)$ "runs away" from this value (in approaches it as one goes to $-\infty$).

Similarly when we solved the logistic equation some equilibria had the property that nearby solutions were attracted to the equilibrium, and some had the property that nearby solutions were repelled away from the equilibrium.

We take this as a definition:

Definition 1. *An equilibrium is said to be* **stable** *if nearby initial points converge to the equilibrium. An equilibrium is* **unstable** *if nearby initial points move away from the equilibrium.*

In the phase line pictures we drew the direction of the arrows tells the story. If the arrows point towards the equilibrium the equilibrium is stable. If the arrows point away from the equilibrium the equilibrium is unstable.

Example 3. *Evaluate the stability of the equilibria in the previous examples:*

Note that the stability of the equilibria alternates: between any two stable equilibria there is an unstable one, and between any two unstable equilibria there is a stable one. Discuss:

It would be nice to have a way to decide whether a given equilibrium is stable or unstable. This is the content of the next theorem

Theorem 1. *Suppose that we have the equation*

$$y' = f(y)$$

and y_0 is an equilibrium ($f(y_0) = 0$). Then

- *If $f'(y_0) > 0$ then the equilibrium is unstable - solutions initially near y_0 move away from y_0 exponentially.*
- *If $f'(y_0) < 0$ then the equilibrium is stable - solutions initially near y_0 move towards y_0 exponentially.*
- *If $f'(y_0) = 0$ further analysis is required.*

The following is not really a proof, but is an argument: Suppose y is initially close to y_0: $y(t) = y_0 + \delta v(t)$

Example 4. Fish

In population biology people often try to understand how various forces can effect populations. For instance, one might try to model the effect of fishing on a fish population by a model of the following form:

$$\frac{dy}{dt} = \underbrace{ky(P_0 - y)}_{logicstic} - \underbrace{h}_{constantharvestrate}$$

Here the first term is the standard logistic term and the second term is a constant, which is meant to represent fish being taken out at a (presumably) constant rate. How does this solution behave?

The righthand side is a quadratic, so there are usually two roots. The quadratic formula gives

$$y = \frac{P_0 \pm \sqrt{P_0^2 - 4h/k}}{/}2$$

so if $\sqrt{h/k}$ is less than $P_0/2$ there are two positive real roots. The lower one has positive derivative and the upper one has negative derivative (WHY??? Draw picture) so the upper one is unstable and the lower one is stable:

Now what happens if $\sqrt{h/k}$ is greater than $P_0/2$?

Another example comes from an acceleration velocity model discussed in the next section

Example 5. *The force of air resistance on a body falling under gravity is assumed to be proportional to the square of the velocity. Find an equation for the position of the falling body. How does it behave?*

$$m\frac{d^2x}{dt^2} = -mg + k\left(\frac{dx}{dt}\right)^2$$