Math 341 Computer Lab - October 21, 2002

In today's lab we will use two features of IODE - the second order equation solver and phase plane plotter - to explore second order linear constant coefficient differential equations.

The first and third sets of exercises use option [3] from the IODE main menu, the second order equation solver, and the second set uses option [2] from the IODE main menu, the phase plane plotter.

Reminders:

To start IODE type **cd IODE** at the unix prompt, to change directory to the IODE main directory. Then type **octave** to start octave running, and finally type **iode** at the octave prompt to load the IODE package.

If you have any trouble you will find the login instructions from the first lab are attached at the end of the lab instructions.

When entering equations remember that IODE uses * for multiplication. An expression like 4y will generate an error: 4*y is the correct way to enter this quantity.

1 The equation:

The equation we will be studying is

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + 4x = 0$$

for different values of the parameter γ.

Exercise 1: Calculate the values of γ for which the above equation is overdamped, critically damped, and underdamped. Recall that this corresponds to the characteristic equation

$$r^2 + \gamma r + 4 = 0$$

having two real roots, one double root, and a complex conjugate pair of roots respectively.

Exercise 2: Plot the solutions to the equation

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + 4x = 0$$

for three values of γ : one for which the equation is underdamped, one for which the equation is critically damped, and one for which the equation is overdamped. You can do this using option [3] Second Order Linear ODEs, from the IODE Main menu. Selecting option [3] will give you the following submenu

```
   (1)x'' + (0.5)x' + (1)x = cos(2*t)

options=(rk2, 0.1)

   [ 1] Enter differential equation
   [ 2] Enter domain and range
   [ 3] Plot numerical solution
   [ 4] Change numerical solver
   [ 5] Plot arbitrary function
   [ 6] Clear plots
   [ 7] Save plot
   [ 8] Quit
```

Option [1] allows you to change the equation, and option [3] will allow you to plot numerical solutions of the equation you've entered. Make sure that you choose $f(t) = 0$. Option [2] is very useful for adjusting the region which is plotted.

Exercise 3abc: In the overdamped case the solution is in the form

$$x(t) = A\exp(r_1 t) + B\exp(r_2 t)$$

and in the critically damped case the solution is in the form

$$x(t) = (a + bt)\exp(r_1 t)$$

Notice that in either case $x(t)$ can only vanish for a single value of t. On the other hand in the underdamped case the solution is given by

$$x(t) = A\exp(rt)\cos(\mu t) + B\exp(rt)\sin(\mu t)$$

which vanishes for infinitely many values of t, since $x(t)$ oscillates between positive and negative values. The purpose of this problem is to explore the question of what happens to these zeros as γ approaches the critical value.

a Speculate on what you think happens to these infinitely many zeros as γ approaches the critical value $\gamma = 4$.

b Use IODE option [3], "Plot Numerical Solution" to plot the solution to

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + 4x = 0$$

with initial conditions $x(0) = 1, x'(0) = 0$ for 4 different values of the damping coefficient: $\gamma = 1$,$\gamma = 2$, $\gamma = 3$, $\gamma = 3.5$. In each case find the locations of the first two zeros of $x(t)$. **Note:** You will need to use option [2], which changes the domain and the range of the graph, to do this problem.

At this point we are done with the second order linear equation solver. You Can select option [8] Quit to return to the IODE main menu.

2 Phase plane.

The goal of the next set of exercises is to explore the idea of the phase plane, which was mentioned briefly in lecture on Friday. Recall that the second order equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + 4x = 0$$

Can be reduced to a pair of first order equations for x, y

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\gamma y - 4x\end{aligned}$$

Exercise 4: Use option [1] Enter functions to plot the phase planes of the above equations for three values of γ - one overdamped, one critically damped, and one underdamped. In each case plot solution curves for several different initial conditions using option [3]. What are the qualitative differences between the solution curves in the overdamped and underdamped cases?

Exercise 5: In the overdamped case the general solution is

$$x(t) = A\exp(r_1 t) + B\exp(r_2 t)$$

and thus

$$y(t) = x'(t) = Ar_1\exp(r_1 t) + Br_2\exp(r_2 t$$

In the special case where one of the constants is zero (to be concrete, we choose $B = 0$) we have

$$\begin{aligned} x(t) &= A\exp(r_1 t) \\ y(t) &= x'(t) = Ar_1\exp(r_1 t) \end{aligned}$$

and thus we have $y = r_1 x$ - the solution curve forms a line in the $x - y$ plane. Similarly when $A = 0$ we have $y = r_2 x$. These special solutions are a simple example of what is called a "stable manifold".

Suppose $\gamma = 5$ Plot the two stable manifolds using Option [3] Plot Numerical Solutions. Think about what $x - y$ values you need to choose to find this special solution.

You will need to plot at least four initial conditions to find the whole stable manifold. In particular you'll need to plot the solution curves for the points $(x = 1, y = r_1)$,$(x = -1, y = -r_1)$,$(x = 1, y = r_2)$,$(x = -1, y = -r_2)$, where r_1 and r_2 are the two roots of the characteristic equation. (Do all these plots in the same color. Red looks nice.)

Exercise 6:

Plot some solution curves through other points (Do this in a different color). Do these solution curves ever cross the stable manifolds? What do you notice about the way these curves approach $(0, 0)$?

3 Inhomogeneous Constant Coefficient Problems

In this section we consider the inhomogeneous problem

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + 4x = \cos(\omega t)$$

for different values of the damping constant γ and the frequency ω. In particular we'll be exploring the idea of resonance - when the driving terms solves the homogeneous problem.

Experiment 7: First consider the case where $\omega = 1$. Use option [2] to set the domain to $(0, 60)$ and the range to $(-1, 1)$. Plot the solution to

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + 4x = \cos(t) \qquad x(0) = 0 \qquad x'(0) = 0$$

for $\gamma = 1$ You should find that the solution approaches a solution which appears sinusoidal with an amplitude of about $1/3$. Repeat this experiment for $\gamma = 0.5$ and $\gamma = 0.25$. Find the limiting amplitude in each case.

Experiment 8: Now repeat the experiment with the equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + 4x = \cos(2 * t) \qquad x(0) = 0 \qquad x'(0) = 0$$

and $\gamma = 1, .5, .25$ Find the limiting amplitude in each case. You may need to play around with the domain and range in order to do this. What conclusions can you draw about the relative sensitivity of the limiting amplitude to the value of γ in the first and second cases.

Aside: In the second case the forcing term is **resonant** when $\gamma = 0$ - it is a solution to the homogeneous problem.

Experiment 9: Plot the solution to

$$\frac{d^2x}{dt^2} + 4x = \cos(2 * t) \qquad x(0) = 0 \qquad x'(0) = 0$$

Produce a graph to show that the amplitude of the solution grows linear in t. This is the phenomenon of resonance. Find the analytical solution in this case.

4 Appendix

Logging In: The login name for the class account is "math341" (all lower case) and the password is mon1pm. Please note that UNIX is *case sensitive.*

If you have a math department account please use that, rather than the class account.

Getting a Window

After you have successfully logged in you will need to open a terminal window. If there is no terminal window open you will need to open one. To do so click on the background with the right mouse button. You should get a menu titled "Workspace Menu". Go to the "Tools" submenu and select "Terminal". This should give you a terminal window.

Finding the IODE directory:

After you have gotten a terminal menu you will need to move to the IODE directory. You can do this by typing "cd iode" at the unix prompt, as follows:

```
u57 1% cd IODE
```

which will **c**hange **d**irectory to the IODE directory.

Note: The aboves assumes that you are sitting at the machine named u57 - your prompt will be slightly different.

Starting Octave/IODE:

The IODE package runs under Octave (a free software package which is largely compatible with Matlab). To start Octave type octave at the unix prompt:

```
u57 2% octave
```

From the Octave prompt simply type "iode" to bring up the iode package"

```
GNU Octave, version 2.0.16 (sparc-sun-solaris2.7).
Copyright (C) 1996, 1997, 1998, 1999, 2000 John W. Eaton.
This is free software with ABSOLUTELY NO WARRANTY.
For details, type ‘warranty’.

octave:1> iode
```

You should get a prompt which closely resembles the following

```
$Id: iode.m,v 1.21 2002-06-03 19:57:31-05 brinkman Exp $
Copyright (c) 2001, Peter Brinkmann (brinkman@math.uiuc.edu)
This program is distributed in the hope that it will be useful,
but WITHOUT ANY WARRANTY; without even the implied warranty of
MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE.  See the
GNU General Public License for more details.

Running under Octave.


Iode Main Menu

  [ 1] Direction fields
  [ 2] Phase planes
  [ 3] Second order linear ODEs
  [ 4] Fourier series
  [ 5] Partial differential equations
  [ 6] Quit

pick a number, any number:
```