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space ↓ time ↓

MATH 553 WAVE EQUATION IN $\mathbb{R}^d \times \mathbb{R}$ (Friday Feb. 16)

$u_{tt} = \Delta u \qquad \Delta u = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

GOAL: SOLVE WAVE EQUATION IN $\mathbb{R}^d$ ($d=2,3$)

Other dimensions given in h.w exercises.

Solve FIRST FOR $d=3$, use this to specialize to $d=2$.

DEFINITION: Given $h(x)$ $x \in \mathbb{R}^n$ define

$$M_h(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} h(x+r\xi)\, dS_\xi$$

← note

Average over sphere of radius r (NOT BALL)

$\omega_n =$ Area of $S^{n-1} = |S^{n-1}|$

$dS_\xi =$ volume element on sphere

Observations: If $h(x)$ is continuous $M_h(x,0) = 0$.

$$\frac{\partial}{\partial r} M_h(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} \vec{\xi} \cdot \vec{\nabla} h(x+r\xi)\, dS_\xi$$

$$= \frac{1}{\omega_n} \int \vec{\nabla} h \cdot \eta \, dS_\xi \qquad \vec{\eta} = \text{outward normal}$$

apply divergence theorem $\int_V \operatorname{div}\vec{f} = \int_{\partial V} \vec{f}\cdot\eta \, dS$

$\frac{\partial}{\partial \xi_i} = r\frac{\partial}{\partial x_i}$

$$\frac{\partial M}{\partial r} = \frac{1}{\omega_n}\int_{|\xi|<1} r\,\Delta_x h \, d\xi$$

$$= \frac{r}{\omega_n}\Delta_x \int_{|\xi|<1} h \, d\xi$$

Now we do change of variables $\vec{\eta} = r\vec{\xi}$

$$\int_{|\xi|<1} h(x+r\xi)\,d\xi = \frac{1}{r^n}\int_{|\eta|<r} h(x+\eta)\,d\eta$$

$$= \frac{\cancel{\omega_n}1}{r^n}\int_0^r \rho^{n-1}\int_{|\xi|=1} h(x+\rho\xi)\,dS_\xi \, d\rho$$

$$= \frac{\omega_n}{r^n}\int_0^r \rho^{n-1} M_n(x,\rho)$$

Thus

$$\frac{\partial M_n}{\partial r} = \frac{1}{r^{n-1}}\Delta_x \int_0^r \rho^{n-1} M_n(x,\rho)\,d\rho.$$

$$r^{n-1}\frac{\partial M}{\partial r} = \Delta_x \int_0^r \rho^{n-1} M_n(x,\rho)\,d\rho$$

$$r^{n-1}\frac{\partial^2 M}{\partial r^2} + (n-1)\, r^{n-2}\frac{\partial M}{\partial r} = \Delta_x\, r^{n-1} M_n(x,\rho)$$

$$\Delta_x M_n(x,r) = \underbrace{\frac{\partial^2 M}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial M}{\partial r}}_{\text{Darboux EQUATION}}$$

NOTE: $\frac{\partial^2 M}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial M}{\partial r}$ is <u>radial</u> <u>part</u> of <u>Laplacian</u>.

This makes sense because M_h is an angular average of h. This "kills" the angular part of the Laplacian.

GENERAL STRATEGY

$$\begin{tikzcd}
\begin{array}{l} u(x,0)=\phi \\ u_t(x,0)=\psi \\ u_{tt}=u_{xx} \end{array} \arrow[r, "\substack{\text{Linear Transformation.} \\ u \mapsto M_u(x,r)}"] \arrow[d, "\text{Solve EQN.}"'] & M_{u(x,0)}(x,r) \arrow[d] \\
u(x,t) & M_{u(x,t)}(x,r) \arrow[l]
\end{tikzcd}$$

This strategy can be applied to solve many linear PDE (and a few nonlinear ones!) We will see this strategy again with the Fourier Transform.

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Application TO WAVE EQUATION:

$$u_{tt} = c^2 \Delta u \qquad \begin{array}{l} u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{array}$$

Take spherical MEAN OF BOTH SIDES

$$\frac{1}{\omega_n} \int_{S_{n-1}} u_{tt}(x + r\xi, t)\, dS_\xi = c^2 \frac{1}{\omega_n} \int \Delta_x u(x + r\xi, t)\, dS_\xi$$

$$\frac{\partial^2}{\partial t^2} M_u = c^2 \Delta_x M_u(x, r, t)$$

$$= c^2 \left(\frac{\partial^2 M_u}{\partial r^2} + \underline{\frac{(n-1)}{r} \frac{\partial}{\partial r}} \right) M_u$$

LIKE 1-D WAVE EQN, but w/ EXTRA TERM!

$$M_u(x, r, 0) = M_\phi(x, r)$$

$$\frac{\partial M_u}{\partial t}(x, r, 0) = M_\psi(x, r).$$

TO RECOVER u we use the fact that

$$\boxed{\lim_{r \to 0} M_u(x, r, t) = u(x, t)}$$

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$n=3$

$$\frac{\partial^2 M}{\partial \tau^2} = \frac{\partial^2 M}{\partial r^2} + \frac{2}{r}\frac{\partial M}{\partial r}$$

multiplying through by r gives

$$\frac{\partial^2 rM}{\partial \tau^2} = r\frac{\partial^2 M}{\partial r^2} + 2\frac{\partial M}{\partial r}$$

$$\frac{\partial^2}{\partial \tau^2}(rM) = \frac{\partial^2}{\partial r^2}(rM)$$ 1-D WAVE EQUATION!

Letting $G = r M_\phi$
$H = r M_\psi$

$$rM = \frac{G(r+c\tau) + G(r-c\tau)}{2} + \frac{1}{2c}\int_{r-c\tau}^{r+c\tau} H(\rho)\,d\rho$$

G, H are ODD (M_ϕ, M_ψ even!) Thus $G(r-c\tau) = -G(c\tau - r)$

$$M = \frac{G(r+c\tau) - G(c\tau-r)}{2r} + \frac{1}{2cr}\int_{c\tau-r}^{c\tau+r} H(\rho)\,d\rho$$ ← EXPLAIN

$$M = \frac{(c\tau+r)M_\phi(x,r,\tau) - (c\tau-r)M_\phi(x,r,\tau)}{2r} + \frac{1}{2cr}\int_{c\tau-r}^{c\tau+r} \rho\, M_\psi(x,\rho)\,d\rho.$$

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Letting $r \to 0$

$$u = \frac{\partial}{\partial \tau}\left(\tau M_g(x,\tau)\right)\Big|_{\tau = ct} + \tau M_h(x, c\tau)$$

$$u = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t \int_{|\xi|=1} \phi(x+ct\xi)\, dS_\xi\right) + \frac{t}{4\pi}\int h(x+ct\xi)\, dS_\xi.$$