Mon Feb 19

WAVE EQN IN $n(=3)$ dimensions.

LAST TIME: A synopsis

$$M_u(x,r) = \frac{1}{\omega_n} * \int_{|\xi|=1} u(x+r\xi)\, dS_\xi$$

$$\omega_n = \text{Area of } S^{n-1} = \int_{|\xi|=1} dS_\xi$$

$$u_{\tau\tau} = c^2 \Delta u \quad \Rightarrow \quad \frac{\partial^2 M_u}{\partial \tau^2} = c^2\left(\frac{\partial^2 M_u}{\partial r^2} + \frac{n-1}{r}\frac{\partial M_u}{\partial r}\right) \; *$$

$u(x,0)=\phi$

$u_\tau(x,0)=\psi$

$$M_u(x,r,0) = M_\phi(x,r)$$

$$\frac{\partial M_u}{\partial \tau}(x,r,0) = M_\psi(x,r)$$

IN $n=3$ $*$ is equivalent to

$$\frac{\partial^2}{\partial \tau^2}(r M_u) = c^2 \frac{\partial^2}{\partial r^2}(r M_u)$$

$$V = r M_u$$

$$V_{\tau\tau} = c^2 V_{rr}$$

$$V(r,x,0) = r M_\phi = G$$

$$V_\tau(r,x,0) = r M_\psi = H$$

ODD Functions

$$V = \frac{G(r+c\tau) + G(r-c\tau)}{2} + \frac{1}{2c}\int_{r-c\tau}^{r+c\tau} H(\rho)\, d\rho.$$

(assume $c\tau > r$)

$$r M_u = \frac{G(r+c\tau) - G(c\tau - r)}{2} + \frac{1}{2c}\int_{c\tau-r}^{c\tau+r} H(\rho)\, d\rho.$$

$$M_u = \frac{G(r+cT) - G(cT-r)}{2r} + \frac{1}{2cr}\int_{cT-r}^{cT+r} H(\rho)\,d\rho.$$

as $r \to 0$

$$u(x;\tau) = M_u(x,0) = \left.\frac{\partial G}{\partial \tau}\right|_{\tau = cT} + H(cT)$$

But
$$G(r) = r\,M_\phi(x,r) \qquad \omega_3 = 4\pi$$
$$H(r) = r\,M_\psi(x,r)$$

$$u = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1}\phi(x+c\tau\xi)\,dS_\xi\right) + \frac{\tau}{4\pi}\int_{|\xi|=1}\psi(x+c\tau\xi)\,dS_\xi$$

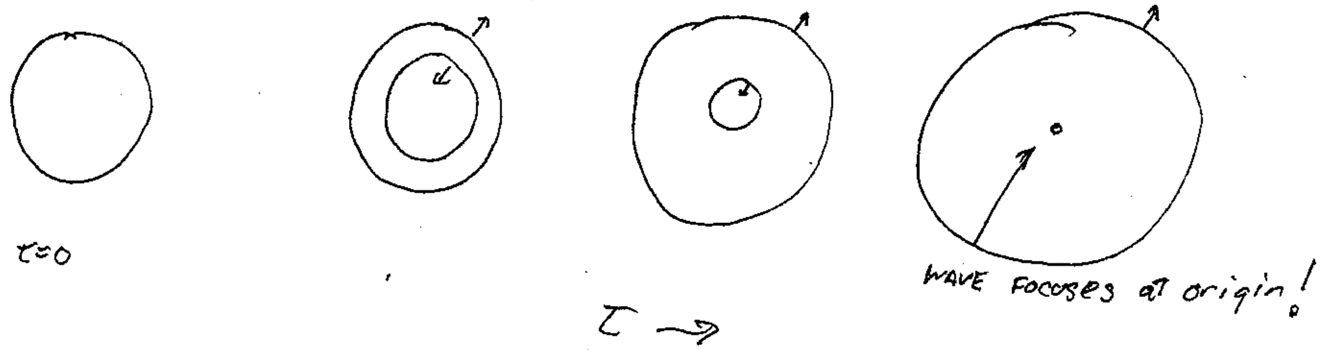
OBSERVATION: If $\phi \in C^3$, $\psi \in C^2$ Then u defined by above is C^2. Loss of 1 Derivative.

NOTE THE LOSS OF REGULARITY: A SOLUTION which has initial data That is C^3 is ONLY guaranteed to be C^2 at later Times. This is part of The geometry of The wave equation in higher dimensions.

CONTRAST w/ D'ALEMBERT

$$u = \frac{1}{2}\big(\underbrace{\phi(x+cT) + \phi(x-cT)}_{C^2}\big) + \frac{1}{2c}\int_{x-cT}^{x+cT}\underbrace{\psi(\rho)}_{C^1}\,d\rho.$$

Picture: Focusing of WAVEFRONTS:



- **SHARP PROPAGATION OF SIGNALS**

u at time T and position x depends ONLY on what happens on sphere centered on point x with radius cT.

Again contrast with D'Alembert

Depends ~~only~~ on ϕ only on sphere

u " ψ on whole ball $|x-\xi| \le CT$

WAVE EQN IN $d=2$. SAME TRICK FAILS

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$v = r^{\frac{1}{2}} u$$

$$\frac{\partial^2 v}{\partial r^2} = r^{\frac{1}{2}}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^{\frac{1}{2}}}\frac{\partial u}{\partial r} - \frac{1}{4} r^{-\frac{3}{2}} u$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial r^2} + \underline{\underline{\frac{1}{4}\frac{1}{r^2} v}}$$

no good!

~~[illegible]~~ IDEA! $2<3$!

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right) \qquad \begin{aligned} u(\vec{x},0) &= \phi(x,y) \\ u_t(\vec{x},0) &= \psi(x,y) \end{aligned}$$

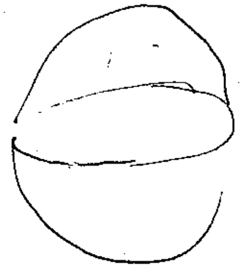
u solves 2-D WAVE EQUATION!

$$u = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t \int_{|\xi|=1} \phi(x+ct\xi_1, y+ct\xi_2)\, dS_\xi \right)$$

$$\psi(\xi_1,\xi_2) = \xi_3 = \sqrt{1-\xi_1^2-\xi_2^2}$$

Surface MEASURE

$$dS_\xi = \sqrt{1+(\psi_{\xi_1})^2+(\psi_{\xi_2})^2}\; d\xi_1\, d\xi_2 = \frac{d\xi_1\, d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}$$



DON'T FORGET 2!

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$$u(x_1,x_2,\tau)= \frac{1}{4\pi}\ \frac{\partial}{\partial\tau}\left(2\tau \int\limits_{\xi_1^2+\xi_2^2<1} \frac{\phi(x_1+c\tau\xi_1,\ x_2+c\tau\xi_2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\, d\xi_1\, d\xi_2 \right)$$

$$+\frac{2\tau}{4\pi} \int\limits_{\xi_1^2+\xi_2^2<1} \frac{h(x_1+c\tau\xi_1,\ x_2+c\tau\xi_2)}{\sqrt{1-\xi_1^2+\xi_2^2}}\, d\xi_1\, d\xi_2$$