MATH 553 LECTURE #2

READING Introduction / Chapt 1.1 McOwen

Transport equation

$$\rho_t + (c\rho)_x = 0$$

Derived from conservation of mass. We may also consider

$$\rho_t + (c\rho)_x = f(x,u,t)$$

where f represents a source term.

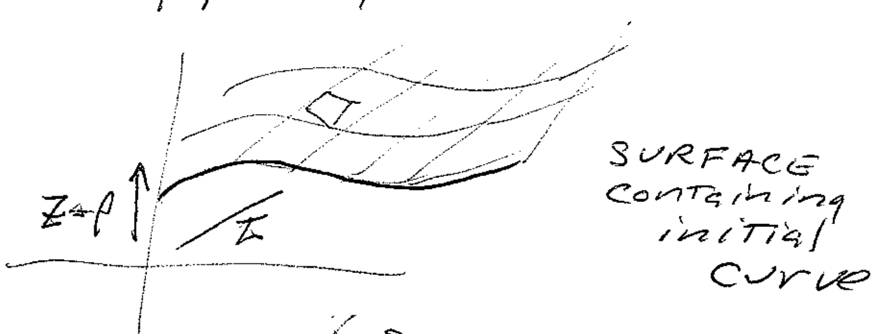
In the constant case c = constant we have the solution

$$\rho(x,t) = f(x-t)$$

Consider the general case

$$a\rho_t + b(\rho,x,t)\rho_x = c(\rho,x,t)$$

Look at the graph $z=\rho(x,t)$

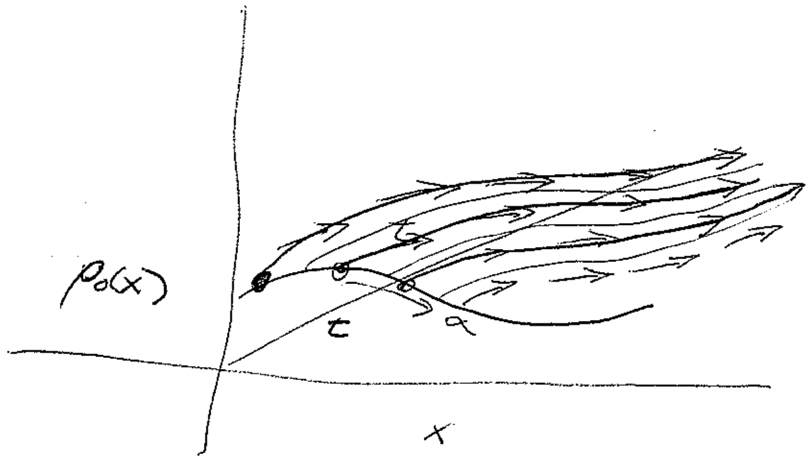


TANgent plane spanned $(1,0,\rho_t)$ & $(0,1,\rho_x)$

normal $(-\rho_t, -\rho_x, 1)$

Consider vector field $\vec{V}(x,y,z) = (a(x,\tau,z),\, b(x,\tau,z),\, c(x,\tau,z))$

$\vec{V}\cdot\vec{n} = 0 \Rightarrow$ vector field lies in tangent plane



Find family of curves tangent to vector field $\vec{V}$

$\vec{X} = \begin{pmatrix} x \\ \tau \\ z \end{pmatrix}$ $\qquad \dfrac{d\vec{X}}{ds} = \vec{V}(\vec{X})$ $\leftarrow$ gives one curve for each initial point

$\vec{X}(0) = (0, \alpha, \rho_0(\alpha))$

E.

parametric representation

$$\begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{vmatrix} 1 & 0 & i \\ 0 & 1 & j \\ \rho_x & \rho_\tau & k \end{vmatrix} \qquad (-\rho_x, -\rho_\tau, 1)$$

$$\left.\begin{array}{l} x = x(\alpha, s) \\ \tau = \tau(\alpha, s) \\ \rho = \rho(\alpha, s) \end{array}\right\}$$ eliminate α, s

$\rho = \rho(x,\tau)$

Example: $\rho_t + \rho_x = 0$ $\qquad \rho(\cancel{t}\,x,0) = \rho_0(x)$

$a = 1$
$b = 1$
$c = 0$

initial curve $\quad \tau = 0$, $x = \alpha$, $z = \rho_0(\alpha)$

$$\frac{d\tau}{ds} = \cancel{a}\,1 \qquad \tau = s*$$

$$\frac{dx}{ds} = 1 \qquad x = s + \alpha$$

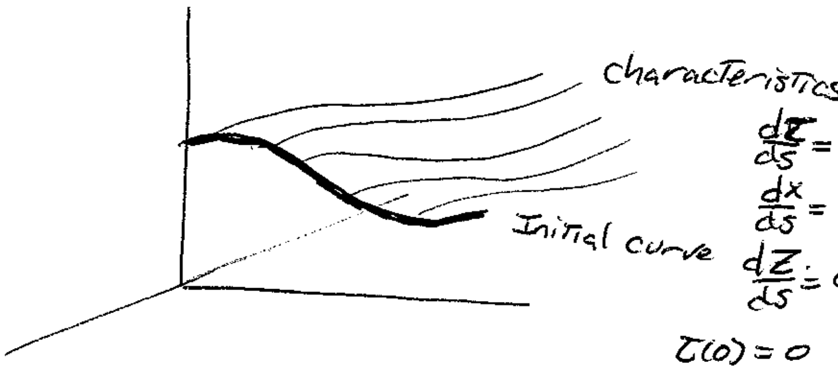
$$\left.\begin{matrix} \tau = s \\ x = s+\alpha \end{matrix}\right\} \quad \alpha = x - \tau$$

$$\frac{dz}{ds} = 0 \qquad z = \rho_0(\alpha) \qquad z = \rho_0(x - \tau).$$

MATH 553 LECTURE #3

EXAMPLES: METHOD OF Characteristics

$$a(x,\tau,u)u_\tau + b(x,\tau,u)u_x \pm c(x,\tau,u)$$
$$u(x,0) = u_0(x)$$



$$\frac{d\tau}{ds} = a(x,\tau,z)$$
$$\frac{dx}{ds} = b(x,\tau,z)$$
$$\frac{dz}{ds} = c(x,\tau,z)$$
$$\tau(0) = 0$$
$$x(0) = \alpha$$
$$z(0) = u_0(\alpha)$$

$$u_\tau + uu_x = 0$$

$$\frac{d\tau}{ds} = 1 \qquad \frac{dx}{ds} = z \qquad \frac{dz}{ds} = 0$$

$$\tau = s \qquad x = zs + \alpha \qquad z = u_0(\alpha)$$
$$= u_0(\alpha)s + \alpha$$

$$u = u_0(x - u\tau) \quad \leftarrow \text{defined implicitly}$$

Solvablility $\quad F(u,x,t) = 0 \quad$ Find $u = u(x,\tau)$ if $\frac{\partial F}{\partial u} \neq 0$

$$\frac{\partial F}{\partial u} = 1 + \tau u_0'(x - u\tau)$$

when does this first vanish? If $u' \geq 0$, Then it never does so. If u_0' is somewhere negative, then it vanishes @

$$\tau = -\frac{1}{u_0'(\alpha)}$$

The first such time is $-\frac{1}{\min u_0'}$

Physically this corresponds to wave breaking or overturning

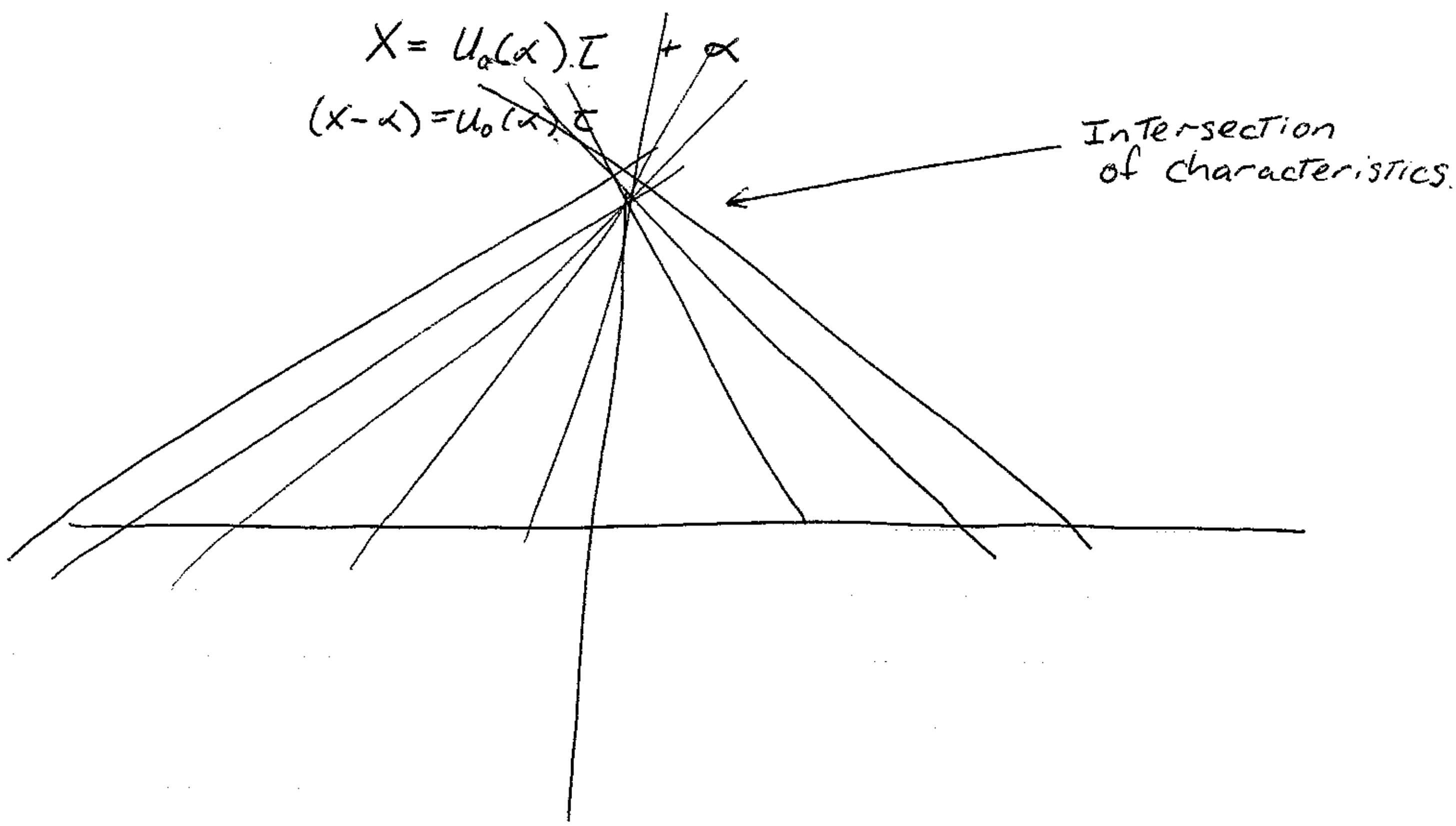
For example if $u_0 = -\tanh x = \frac{-\sinh x}{\cosh x}$

$u_0' = -1 + \frac{\sinh^2}{\cosh^2} = \frac{-1}{\cosh^2}$

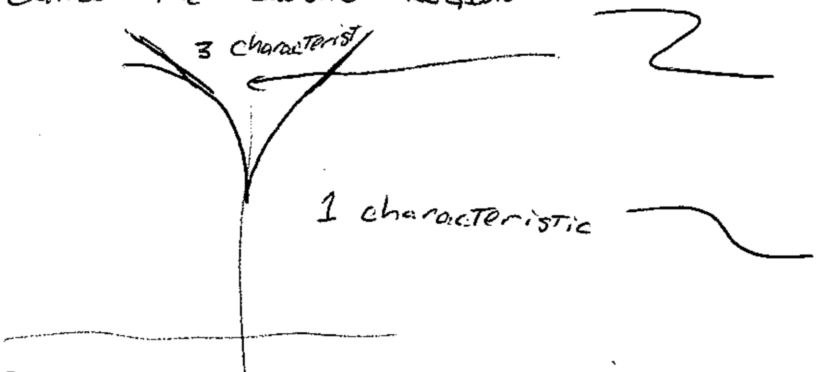
$\min_x u_0' = -1 \quad (@\, x=0)$

time of first breaking is $\tau = 1$

PICTURE Characteristics are a one parameter family of lines parameterized by α



THE REGION WHICH IS (TRIPLY) COVERED by characteristics is called The "CAUSTIC" REGION



NOTE: FROM POINT OF VIEW OF A SURFACE THERE IS NOTHING WRONG

LATER WE'LL DISCUSS HOW TO "REGULARIZE" SOLUTION.

CLOSED CHARACTERISTICS

Example: $x\,\partial_y u - y\,\partial_x u = u \qquad u(x,0) = u_0(x) \qquad x \geq 0$

characteristics

$$\frac{\partial y}{ds} = x \qquad \frac{\partial x}{\partial s} = -y$$

$$\frac{\partial z}{\partial s} = z$$

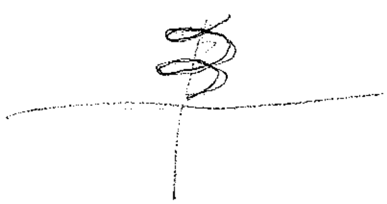
Data specified

$$\frac{d^2 y}{ds^2} = \frac{dx}{ds} = -y \qquad \begin{aligned} y &= A \sin s + B \cos s \\ x &= A \cos s - B \sin s \end{aligned}$$

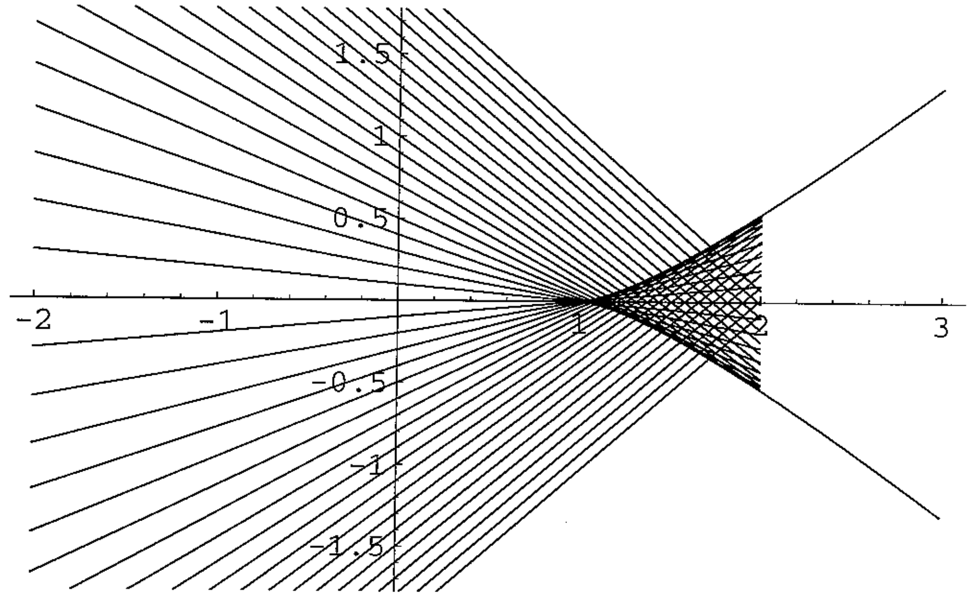
$$\begin{aligned} y(0) &= 0 \\ x(0) &= \alpha \end{aligned} \qquad \begin{aligned} y &= \alpha \sin s \\ x &= \alpha \cos s \end{aligned} \qquad \alpha = \sqrt{x^2+y^2} = r$$

$$\frac{dz}{ds} = z \Rightarrow z = u_0(\alpha)\, e^{s}$$

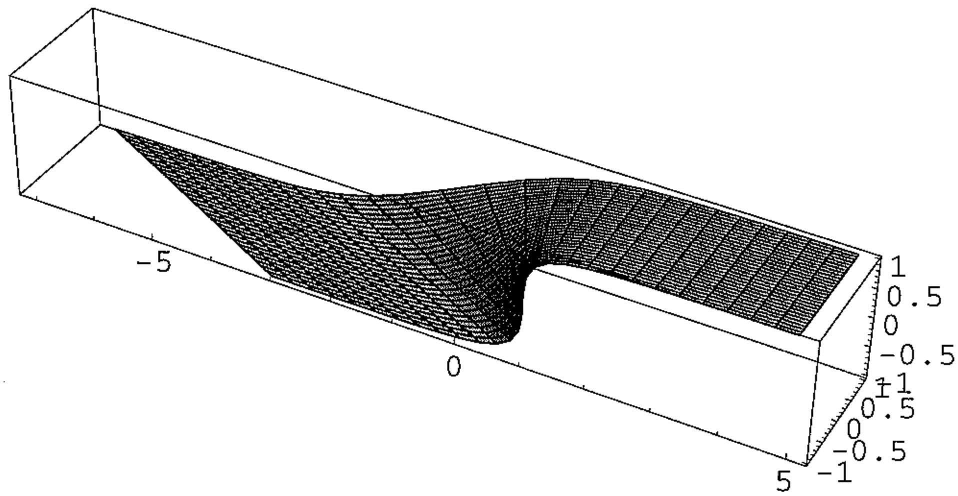
$u = u(r)\, e^{\theta}$ ← NOT single valued!



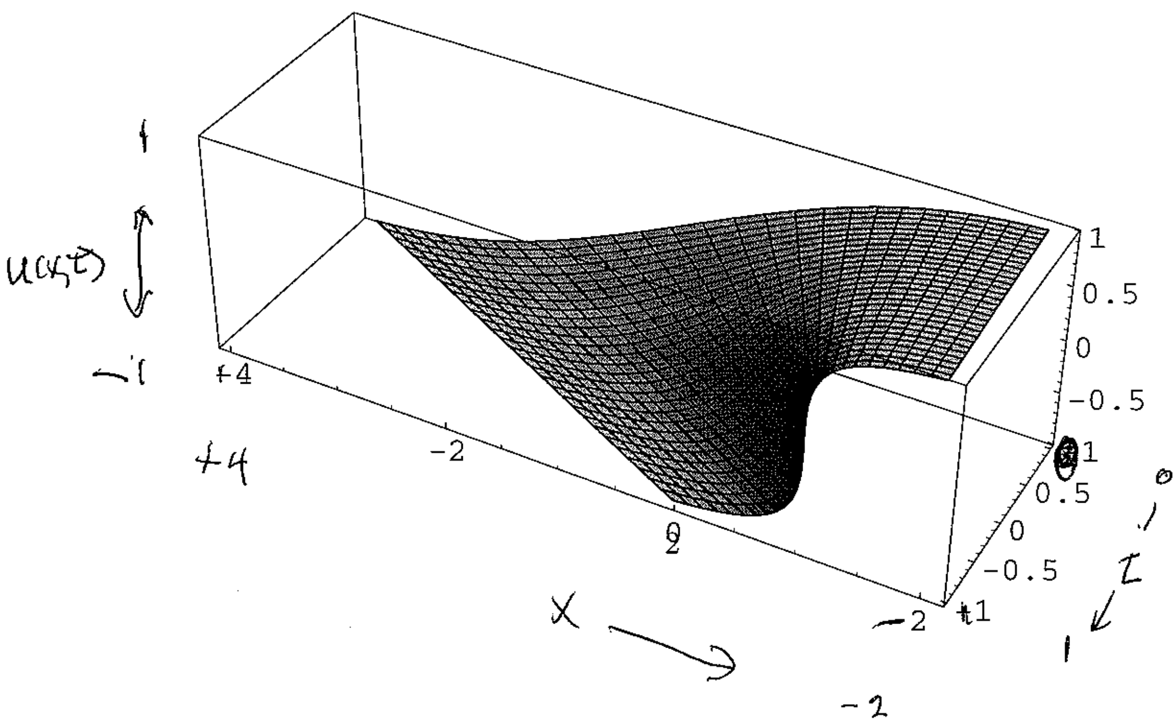
Exponential Helix



1.5
1
0.5
-0.5
-1
-1.5
-2
-1
1
2
3



-5
0
5
1
0.5
0
-0.5
-1
0.5
0
-0.5
-1



1
u(x,t)
-1
+4
-2
0
2
-2
1
0.5
0
-0.5
-1
0.5
0
-0.5
1
x
t
0
1
-2

The top figure shows the characteristics for the Burgers equation with $u_0(x) = -\tanh(x)$ initial data. Note for $t < 1$ the $x - t$ plane is singly covered by characteristics, while at $t = 1$ a region which is triply covered by characteristics is formed.

The second and third figures show the graph of $u(x, t)$ in the $x - t$ plane. Note the steepening of the curve and the formation of infinite slope at $t = 1$.