MATH 553 LECTURE # 4 WEAK SOLUTIONS

In the last lecture we saw the solution to the inviscid Burger's equation

$$u_\tau + u u_x = 0 \qquad u(x,0) = u_0(x)$$

is given parametrically by

$$\tau = s$$
$$x = u_0(\alpha) s + \alpha$$
$$u = u_0(\alpha)$$

From which we can eliminate τ, s to find

$$u = u_0(x - u\tau)$$

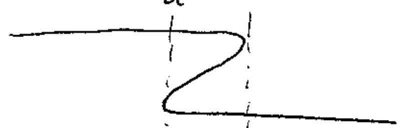
This defines $u = u(x,\tau)$ uniquely for short times, though not necessarily for all times. Last time we saw this through the implicit function theorem. Another way to see this (which is basically equivalent) is to look at the Jacobian

$$J = \begin{vmatrix} \frac{\partial \tau}{\partial s} & \frac{\partial x}{\partial s} \\ \frac{\partial \tau}{\partial \alpha} & \frac{\partial x}{\partial \alpha} \end{vmatrix} = \begin{vmatrix} 1 & u_0(\alpha) \\ 0 & 1 + u_0'(\alpha) s \end{vmatrix}$$

which vanishes @ $s = \tau = -\frac{1}{u_0'}$

It is useful in many situations to assign a meaning to the solution after the time at which the solution ceases to be single valued. There is a nice way to do this, however solutions are not unique and one must use some procedure to pick a solution.

Suppose that $u_t + u u_x = 0$ has a solution



at some fixed time t_1 we could make this a function in the following way

However this raises an obvious question: How do we choose x_0? In principle any point $x_0 \in [a,b]$ would work.

The usual way to choose x_0 is to make use of the fact that Burger's equation is a <u>conservation law</u>: we derived it from the physical requirement that

$$\frac{d}{dt}\int_a^b u\,dx = -cu\Big|_{x=a}^{x=b}$$

(Burger's is special case $c = \frac{u}{2}$

or

$$\int_a^b \left(u_t + (cu)_x\right) dx = 0$$

If The integrand is continuous we can show That

$$u_t + (cu)_x = 0$$

but The integrated form makes sense much more generally. We can use This To establish how To choose $x_0(t)$.

RANKINE-Hugoniot Jump condition: Suppose $u(x,t)$ has a jump @ $x_0(t)$. How should $x_0(t)$ be chosen To satisfy The conservation law (in The integrated form)

$u(x_0^+) = u_{Right}$

$u(x_0^-) = u_{left}$

$$\text{Total mass} = \int_a^b u\,dx = \int_a^{x_0(t)} u\,dx + \int_{x_0(t)}^b u\,dx$$

$$\frac{dM}{dt} = x_0'(t)\Big(u(x_0^-(t),t) - u(x_0^+(t),t)\Big) + \int_a^b u_t$$

$$\int_a^{x_0^-(t)} (cu)_x + \int_{x_0^+}^b (cu)_x = cu(b) - cu(x_0^+) + cu(x_0^-) - cu(a)$$

So for Mass to be conserved across The jump we need

$$x_0'\left(u_{left} - u_{right}\right) + \left(cu_{left} - cu_{Right}\right)$$

$$x_0' = \frac{c(u_{left})\,u_{left} - c(u_{Right})\,u_{right}}{u_{left} - u_{right}}$$

Friday Jan 29   Quasilinear EQNS; WEAK SOLNS

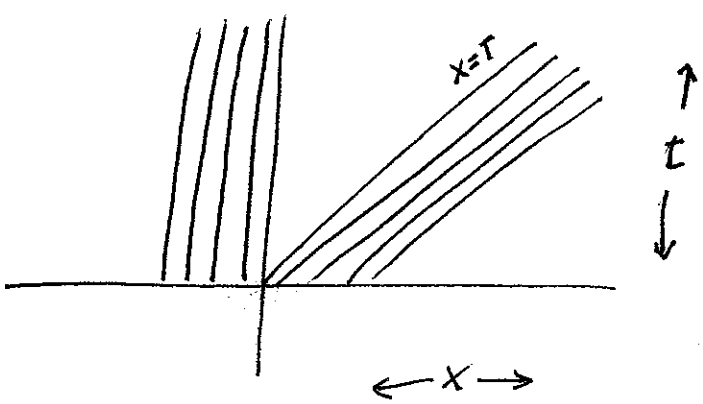
WEAK SOLUTIONS: PART 2   Rarefaction fans

$$u_\tau + u u_x = 0 \qquad u_0(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\frac{dx}{ds} = z \quad x(0) = \alpha \qquad x = u_0(\alpha) s + \alpha$$

$$\frac{d\tau}{ds} = 1 \quad \tau(0) = 0 \qquad \tau = s$$

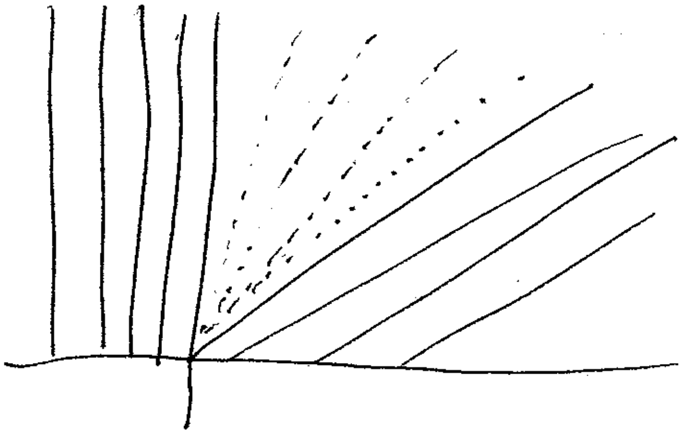
$$\frac{dz}{ds} = 0 \quad z(0) = u_0(\alpha) \qquad z = u_0(\alpha)$$



NOTE THAT THE REGION $\tau > 0$, $x \in (0,\tau)$ is not COVERED BY CHARACTERISTICS.

TO DEAL WITH THIS WE "INVENT" A FAMILY OF CHARACTERISTICS TO FILL THE WEDGE $\tau > 0$ $x \in (0,\tau)$

$x = \eta \tau \qquad \eta \in (0,1)$



If we look for a function constant along characteristics

$$u = f(\eta) = f\left(\frac{x}{T}\right)$$

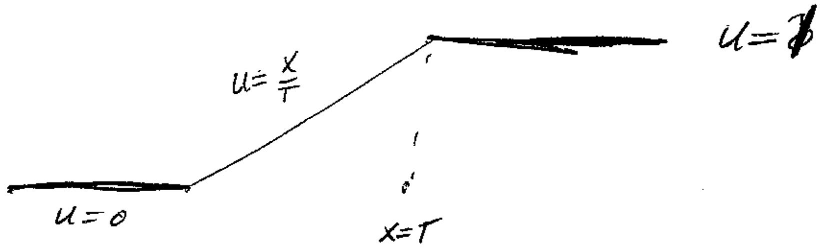
$$u_T = -\frac{x}{T^2} f'\left(\frac{x}{T}\right) \qquad u_x = \frac{1}{T} f'\left(\frac{x}{T}\right)$$

$$u_T + u u_x = f'\left(\frac{x}{T}\right)\left(\frac{f\left(\frac{x}{T}\right)}{T} - \frac{x}{T^2}\right) = 0$$

$f' = 0$ ($f =$ constant) OR

$$f\left(\frac{x}{T}\right) = \frac{x}{T}$$

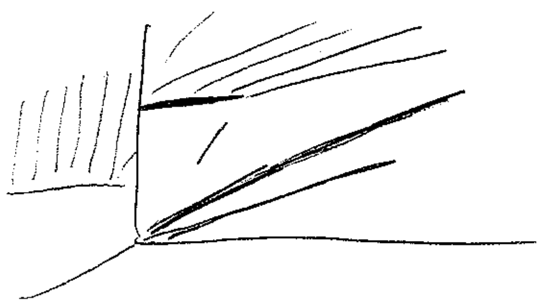
Note that this can be used to piece together solutions:



This type of solution is called a rarefaction fan.

(NOTE:

More geometric picture



Instead of discontinuous function consider continuous curve.

Quasilinear EQUATIONS LECTURE 6

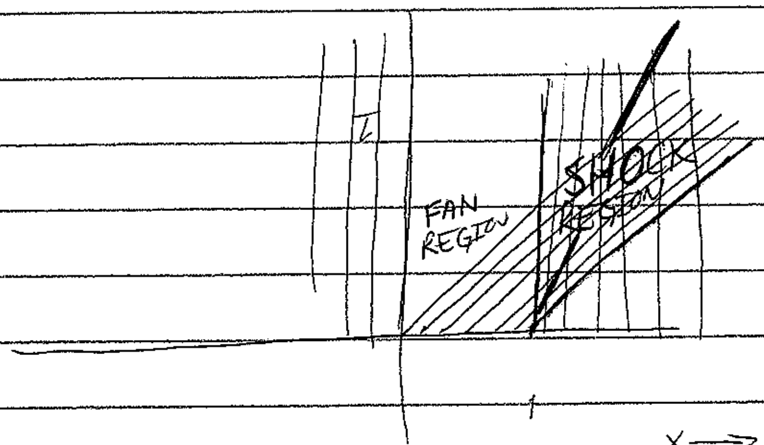
ENVELOPES

EXAMPLE: $u_\tau + u u_x = 0$ $\quad u(x,0) = \begin{cases} 0 & x<0 \\ 1 & x\in(0,1) \\ 0 & x>1 \end{cases}$

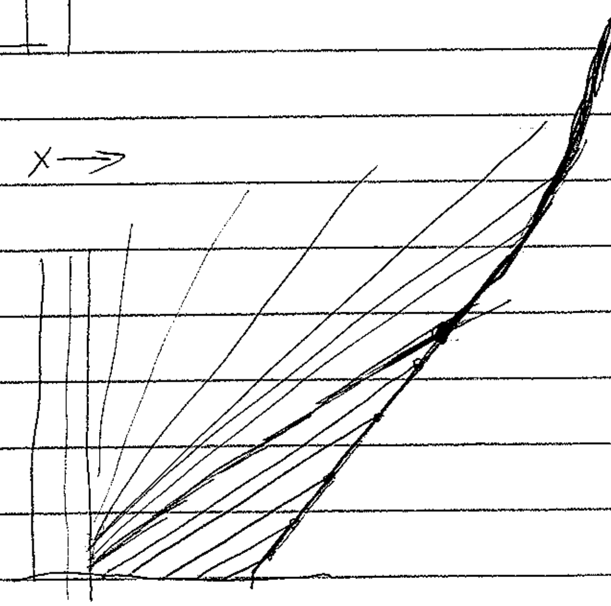
$\tau = s$

$x = u_0(\alpha)\tau + \alpha$

$z = u = u_0(\alpha)$



$$u = \begin{cases} 0 & x<0 & \text{(i)} \\ \frac{x}{\tau} & x\in(0,\tau) & \text{(ii)} \\ 1 & x\in(\tau, 1+\frac{\tau}{2}) & \text{(iii)} \\ 0 & x > 1+\frac{\tau}{2} & \text{(iv)} \end{cases}$$



NOTE: AT Time $\tau = 2$ REGION (iii) vanishes. What happens then? Fan Hits shock! This changes the shock propagation speed!

Before: $u_{left} = 1 \quad u_{right} = 0$

$$\frac{dx}{d\tau} = \frac{1}{2}\cancel{(u_{left} + u_{right})} = \underline{\frac{1}{2}}$$

After $\quad u_{left} = \frac{x}{\tau} \quad u_{right} = 0$

$$\frac{dx}{d\tau} = \frac{1}{2}\frac{x}{\tau}$$

$$\frac{dx}{x} = \frac{1}{2}\frac{d\tau}{\tau}$$

$$\ln x = \frac{1}{2}\ln\tau + c$$

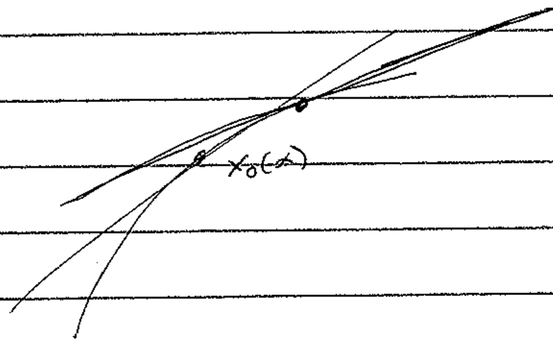
$$x = c\tau^{\frac{1}{2}}$$

$$x(2) = 2 \Rightarrow x = \sqrt{2\tau}$$

Envelopes: Suppose I have a one-parameter family of curves

$$y = f(x, \alpha)$$

I'd like to find an <u>envelope</u> of these curves: A curve $y = g(x)$ with the property that the curves $f(x, \alpha)$ are <u>tangent</u> to g at points where they intersect g.



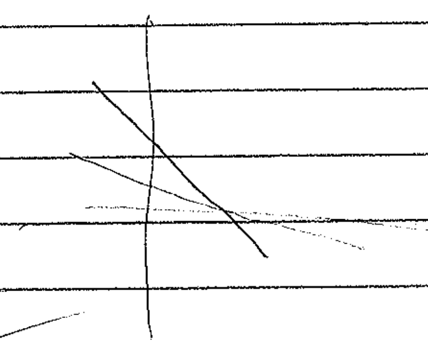
$$f(x_0+\Delta x_0', \alpha) = f(x_0,\alpha) + f_x(x_0,\alpha)\,\Delta\alpha\, x_0' + f_\alpha(x_0,\alpha)\,\Delta\alpha$$

$$g(x_0+\Delta x_0) = g(x_0) + \Delta\alpha\, x_0'\, g_x$$

TANGENCY IMPLIES $g_x = f_x \Rightarrow f_\alpha(x_0,\alpha) = 0$

$\therefore \left.\begin{array}{l} y = f(x,\alpha) \\ \frac{\partial f}{\partial \alpha}(x,\alpha) = 0 \end{array}\right\}$ defines Envelope.

Example $(1-\alpha)y + \alpha x = 1$



$y = -\frac{\alpha x}{1-\alpha} + \frac{1}{1-\alpha}$

$y_\alpha = \left(\frac{-1}{1-\alpha} + \frac{-\alpha}{(1-\alpha)^2}\right)x + \frac{1}{(1-\alpha)^2}$

Example $\cos\theta\, y + \sin\theta\, x = 1$

$y = -\tan\theta\, x + \frac{1}{\cos\theta}$

$y_\theta = -\frac{x}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta} \qquad x = \sin\theta$

$\cos\theta = \sqrt{1-x^2}$

$$y = \frac{1 - x\sin\theta}{\cos\theta} = \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

SEMICIRCLE

Fully Nonlinear EQNS

$$* \quad F(x, y, u, u_x, u_y) = 0 = F(x, y, z, p, q)$$

(x_0, y_0, u_0)

SUPPOSE we KNOW $u_x(x_0, y_0) = p_0 \quad u_y(x_0, y_0) = q_0$

$$(z - z_0) = p_0(x - x_0) + q_0(y - y_0)$$

EQUATION FOR TANGENT PLANE. <u>HOWEVER</u> DETERMINING this is part of the problem.

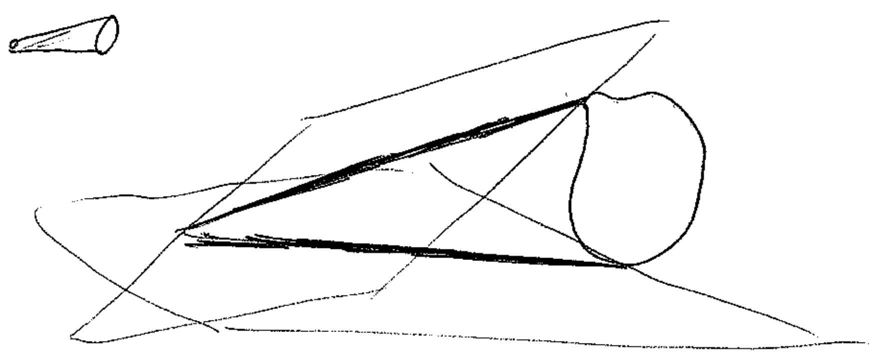
USING EQN * WE CAN (IN principle) FIND $p = p(q)$

Family of Tangent PLANES $\quad (z - z_0) = p(q)(x - x_0) + q(y - y_0)$

Family of Tangent planes defines an Envelope called Monge cone DEFINED BY

$$(z - z_0) = p(q)(x - x_0) + q(y - y_0)$$

$$\frac{\partial p}{\partial q}(q)(x - x_0) + (y - y_0) = 0$$



TANGENT PLANES ARE TANGENT TO Monge cone along a line.

EQNS: $$dz = p\,dx + q\,dy$$
$$0 = dx + \frac{\partial q}{\partial p}\,dy$$

BY IMPLICIT function theorem
$$F_p + \frac{\partial q}{\partial p} F_q = 0.$$

$$0 = F_q\,dx - F_p\,dy$$

$$\underline{\frac{dx}{d\tau} = F_p} \qquad \frac{dy}{d\tau} = F_q. \qquad \frac{dz}{d\tau} = p\frac{dx}{d\tau} + q\frac{dy}{d\tau}$$
$$= pF_p + qF_q.$$

Now we need EQNS FOR $\frac{dp}{d\tau}, \frac{dq}{d\tau}$

$$F_x + u_x F_z + u_{xx} F_p + u_{xy} F_q = 0$$
$$F_y + u_y F_z + u_{xy} F_p + u_{yy} F_q = 0$$

$$\frac{dp}{d\tau} = x_\tau u_{xx} + y_\tau u_{xy} = -\left(F_x + u_x F_z\right) = -\left(F_x + pF_z\right)$$
$$\frac{dq}{d\tau} = -\left(F_y + qF_z\right)$$

Initial conditions   Parameterize initial curve:

$$u(x(\alpha), y(\alpha)) \equiv u_0(\alpha)$$

$$x(0) = x_0(\alpha)$$
$$y(0) = y_0(\alpha)$$
$$z(0) = u_0(\alpha)$$

Two additional conditions:

$$\frac{dx_0}{d\alpha} u_x + \frac{dy_0}{d\alpha} u_y = u_0'(\alpha) \quad ($$

$$\frac{dx_0}{d\alpha} p_0(\alpha) + \frac{dX_0}{d\alpha} q_0(\alpha) = u_0'(\alpha)$$

$$F(x_0(\alpha), y_0(\alpha), u_0(\alpha), p_0(\alpha), q_0(\alpha)) = 0.$$

Solve for $p_0(\alpha), q_0(\alpha)$!