On the pagenumber of k-trees

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Abstract

A *p-page embedding* of G is a vertex-ordering π of $V(G)$ (along the "spine" of a book) and an assignment of edges to p half-planes (called "pages") such that no page contains crossing edges. The *pagenumber* of G is the least p such that G has a p-page embedding. We disprove a conjecture of Ganley and Heath by showing that for all $k \geq 3$, there are k-trees that do not embed in k pages. On the other hand, we present an algorithm that produces k-page embeddings for a special class of k-trees.

1 Introduction

The pagenumber (or book thickness) of a graph G was introduced by Bernhart and Kainen [1]. Given a graph G, a *p-page embedding* of G is a vertex ordering π of $V(G)$ (along the "spine" of a book) and an assignment of edges to p half-planes (called "pages") such that no page contains crossing edges. Equivalently, each page consists of an outerplanar embedding of a subgraph of G having the vertices ordered according to π on the unbounded face. These subgraphs decompose G. The pagenumber of G, denoted $\mathrm{bt}(G)$, is the minimum p such that G has a p-page embedding. We say that G "embeds in p pages" when $\mathrm{bt}(G) \leq p$.

Note that $\mathrm{bt}(G) = 1$ if and only if G is outerplanar. Bernhart and Kainen [1] observed that $\mathrm{bt}(G) \leq 2$ if and only if G is a subgraph of a Hamiltonian planar graph. Pagenumber has been studied on several classes of graphs, including planar graphs [9], graphs with genus g [5, 6] and complete bipartite graphs [3, 7]. In this paper, we study pagenumber of k-trees.

Among several equivalent definitions of k-trees, the inductive definition is convenient for our arguments. A *k-tree* is either the complete graph K_k or a graph obtained from a k-tree G by adding one vertex whose neighborhood is a k-clique in G (a *k-clique* is a set of k pairwise adjacent vertices). The 1-trees are simply the trees, which are outerplanar, and hence they

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have pagenumber 1. Chung, Leighton, and Rosenberg [2] showed that the pagenumber of every 2-tree is at most 2. Ganley and Heath [4] exhibited k-trees that require k pages and proved that if G is a k-tree, then $\mathrm{bt}(G) \leq k+1$. They conjectured that every k-tree embeds in k pages; we disprove this conjecture.

Theorem 1. *For $k \geq 3$, there is a k-tree that does not embed in k pages.*

First, we present an algorithm that embeds many k-trees in k pages, using tree-decompositions of graphs. Let $G[X]$ denote the subgraph of G induced by vertex set X. A *tree-decomposition* of a graph G consists of a host tree T and a family $\{X_i\colon\ i \in V(T)\}$ of subsets of $V(G)$ (called *bags*, perhaps originally by Bruce Reed) such that (1) $G = \bigcup_{i\in V(T)} G[X_i]$ and (2) for each $v \in V(G)$, the set $\{i\colon\ v \in X_i\}$ induces a subtree of T. We use $(T, \mathbf{X})$ to denote a tree-decomposition in which $\mathbf{X}$ is the set of bags.

The *width* of a tree-decomposition $(T, \mathbf{X})$ is $\max_{i\in V(T)}\{|X_i| - 1\}$. The *treewidth* of G is the minimum width among all tree-decompositions of G. (Since every graph has a tree-decomposition with all vertices in one bag, treewidth is well-defined.) A tree-decomposition of width k is *smooth* if the bags for any two adjacent vertices of the host tree have k common elements. By the inductive definition, a k-tree has a smooth tree-decomposition such that every bag is a $(k + 1)$-clique.

Togasaki and Yamazaki [8] showed that if G is a k-tree and G has a smooth tree-decomposition whose host tree is a path, then $\mathrm{bt}(G) \leq k$. We enlarge the family of k-trees for which the conclusion holds.

Theorem 2. *If a k-tree G has a smooth tree-decomposition with width k such that the host tree has maximum degree at most 3, then $bt(G) \leq k$.*

The k-tree we construct in Theorem 1 has a smooth tree-decomposition whose host tree has maximum degree $k + 2$. This leaves open the question of finding the maximum D such that every k-tree having a smooth tree-decomposition whose host tree has maximum degree at most D has a book embedding in k pages. We have shown that $3 \leq D < k + 2$.

2 Construction of k-Page Embeddings

We provide an algorithm that produces a k-page embedding of a k-tree G from a smooth tree-decomposition $(T_0, \mathbf{X}_0)$ of G in which T_0 has maximum degree at most 3.

Since the members of $\mathbf{X}_0$ correspond bijectively to the vertices of T_0, we refer to the bags *as* vertices of T_0. Choose a leaf bag $\{a_1, \ldots, a_{k+1}\}$ of T_0; it will be convenient to name this bag A_{k+1}. Note that exactly one vertex of A_{k+1} does not appearing in the neighbor of A_{k+1} in T_0; index the elements of A_{k+1} so that this vertex is a_{k+1}.

In T_0, each bag X is reached by exactly one path from A_{k+1}. Since $(T_0, \mathbf{X}_0)$ is smooth, X contains exactly one vertex that does not appear in any vertex of this path other than X. For each bag X_i, we let x_i denote this distinguished vertex.

Conversely, since G is connected, every vertex outside A_{k+1} appears in exactly one closest bag to A_{k+1} and is the distinguished vertex for that bag. To have every vertex of G be the distinguished vertex for some bag, we modify T_0 by adding a path $\langle A_1, \ldots, A_k \rangle$ with $A_i = \{a_1, a_2, \ldots, a_i\}$ and A_k adjacent to A_{k+1}. Let T denote the new tree, and let $\mathbf{X} = \mathbf{X}_0 \cup \{A_1, \ldots, A_k\}$; now $(T, \mathbf{X})$ is a tree-decomposition of G.

We refer to vertex A_1 as the root of T. Viewed from A_1, the distinguished vertex for each A_i is a_i. The new tree-decomposition $(T, \mathbf{X})$ is not smooth, but the k added bags with their distinguished vertices simplify the presentation of the proof. The vertices of G now correspond bijectively to the bags. For $x \in V(G)$, we refer to the bag whose distinguished vertex is x as $\overline{x}$; when the context is clear we write X for $\overline{x}$.

While exploring T from the root, the algorithm uses this bijection from $V(G)$ to $V(T)$ to produce a vertex ordering and a k-edge-coloring of G so that the endpoints of two edges with the same color do not occur alternately in the vertex ordering. Such an ordering and coloring define a k-page embedding. The idea is to use the correspondence between vertices and bags to color the edges of T using $k+1$ colors, and then use the edge-coloring of T to produce the k-edge-coloring of G.

In a graph, a *u, v-path* is a path from u to v. We say that X is an *ancestor* of Y and Y is a *descendant* of X if X lies on the A_1, Y-path in T. We will use the following statement about the relationship between G and T to define the edge-coloring of G.

Lemma 3. *If $xy \in E(G)$, then X is an ancestor of Y or Y is an ancestor of X in T.*

Proof. If $xy \in E(G)$, then x and y must appear in some common bag; since the bags containing a vertex of G induce a subtree of T, every bag in the X, Y-path in T contains x or y. Note also that x does not appear in any bag that is an ancestor of X in the rooted tree T. The claim follows. □

We refer to the subtrees of T rooted at the left and right children of X as the (left and right) *subtrees of X*.

2.1 The algorithm

First we produce the vertex ordering π from T. Initialize π to (a_1). Begin a breadth-first search of T from bag A_1. Designate the child(ren) of a bag X in T as its left-child or right-child, arbitrarily. When searching from bag X, having already assigned vertex x a position in π, place the vertex corresponding to its left child (if it has one) immediately before x in π and the vertex corresponding to its right child (if it has one) immediately after x in π. The vertices for bags in the left subtree of X comprise a consecutive segment immediately before x under π, and those corresponding to the right subtree of X comprise a consecutive segment immediately after x under π.

For a bag $Y \in V(T) - \{A_1, \ldots, A_{k+1}\}$ with parent X, recall that $|X - Y| = 1$ and that $\overline{X - Y}$ denotes the bag associated with the vertex of $X - Y$. When Z is an ancestor of Y, we use $Z : Y$ to denote the edge incident to Z on the Z, Y-path in T.

Define a $(k+1)$-coloring f of $E(T)$ as follows. For each edge in T, one endpoint is the parent of the other. When X is the parent of Y in T, let

$$f(XY) = \begin{cases} j, & \text{if } XY = A_jA_{j+1}; \\ k+1, & \text{if } X \notin \{A_1, \dots, A_k\} \text{ and } \overline{X-Y} = X; \\ f(\overline{X-Y} : Y), & \text{if } X \notin \{A_1, \dots, A_k\} \text{ and } \overline{X-Y} \neq X. \end{cases}$$

We use f to define a $(k+1)$-coloring g of the edges of G. If $xy \in E(G)$, then by Lemma 3, we may assume by symmetry that X is an ancestor of Y. Define $g(xy) = f(X:Y)$.

2.2 Validity of the algorithm

First we show that g uses only the colors 1 through k.

Lemma 4. *No edge in G is assigned color $k+1$ under g.*

Proof. The color $g(xy)$ is the color on an edge in T. Since $g(xy) = f(X:Y)$, we have $g(xy) = f(XZ)$, where Z is the child of X on the X,Y-path in T. If $f(XZ) = k+1$, then the definition of f implies that x appears in no bag in the subtree of X that contains Z, and thus x and y could not appear in a bag together and could not form an edge. □

For colors other than $k+1$, we think of the color on an edge from X to a child of it in T as the color *associated with* x in the subtree rooted at that child. For such an edge XY, let w be the unique vertex of $X-Y$. When $f(XY) \neq k+1$, the value $f(XY)$ is the color associated with w in the subtree of W that contains XY, by the definition of f.

Lemma 5. *If X is an ancestor of Y such that $x \in Y$, then the color j associated with x in the subtree of X that contains Y does not appear on any edge of the X,Y-path in T except the initial edge $X:Y$.*

Proof. Consider a bag X closest to A_1 in T at which the claim fails. We have $j \leq k$, since otherwise $x \notin Y$, as observed in the proof of Lemma 4. Note that $j = f(X:Y)$. If j appears again on the X,Y-path, then let ZZ' with parent Z be the edge on which it first reappears. Since j reappears on ZZ', the vertex Z cannot be A_j. Hence the definition of f yields $f(ZZ') = f(W:Z')$, where $\{w\} = Z - Z'$. Hence $w \notin Y$; since $x \in Y$, we have $x \neq w$. We conclude that W is an ancestor of X, since ZZ' was the first reappearance of j. Now j is the color associated with w in the subtree of W that contains Z, and $w \in Z$. This contradicts the choice of X as the failure closest to A_1. □

Proof of Theorem 2. By Lemma 4, g is a k-edge-coloring of G. It remains to show that g does not give the same color to edges whose endpoints alternate in π. Let xy and uv be such edges. By Lemma 3, we may assume that X is an ancestor of Y and U is an ancestor of V. Since the algorithm is symmetric with respect to left and right, we may also assume that Y is in the right subtree of X, and hence $\pi(x) < \pi(y)$. Recall that $g(xy) = f(X:Y)$.

We show that $g(uv) \neq g(xy)$. Since the right subtree of X is listed immediately after X under π and the edge uv crosses the edge xy, the right subtree of X must contain U or V.

Suppose first that U is in the right subtree of X. This implies that V is also in the right subtree of X, since U is an ancestor of V.

If V is in the left subtree of U, then $\pi(x) < \pi(v) < \pi(y) < \pi(u)$. Since the vertices of this subtree appear just before U in the ordering, Y also must be in the left subtree of U. Thus U lies along the X, Y-path in T, and by Lemma 5 the color $g(xy)$ associated with X in its right subtree cannot be the same as the color $g(uv)$ associated with U in its left subtree.

On the other hand, if V is in the right subtree of U, then $\pi(x) < \pi(u) < \pi(y) < \pi(v)$, and we see that Y is also in the right subtree of U. Again, U lies along the X, Y-path in T, and Lemma 5 again yields $g(uv) \neq g(xy)$.

Finally, if U is not in the right subtree of X, then V must be. Since U is an ancestor of V but is not in the right subtree of X, it must be an ancestor of X. Now X lies along the U, V-path in T. By Lemma 5, we conclude that $g(uv) \neq g(xy)$. Therefore, our coloring g together with our ordering π yields a valid book embedding of G in k pages. □

Given the smooth tree-decomposition used by the algorithm, the computations by which the algorithm produces the k-page embedding can easily be implemented to run in constant time per edge. Since k is fixed, this is linear in the number of vertices.

3 A k-Tree With No k-Page Embedding

We construct a k-tree G that does not embed in k pages. Given any ordering of $V(G)$, we use pigeonholing arguments to produce an induced subgraph of G that cannot be embedded in k pages under that ordering. This suffices, since a k-page embedding of G contains a k-page embedding of every induced subgraph.

The graph G has a central k-clique X with vertices $x_1, \ldots, x_k$. Next we add vertices $y_1, \ldots, y_{kN}$, where $N = (k^2 + k + 5)$, each adjacent to all of X. Finally, we add many vertices, called *children*, each adjacent to $k - 1$ vertices in X and one y_i. A child has *type* (i, j) if it is adjacent to y_i and nonadjacent to x_j. There are k^2N different types of children. We create $3k(Nk + k + N)$ children of each type, so G altogether has $3k^3N(Nk + k + N)$ children. We refer to all children adjacent to vertex x_i (or y_i) as the *children of* x_i (or y_i).

Fix a circular ordering π of $V(G)$; we will show that G has no k-page embedding under π. By the Pigeonhole Principle, there are at least N vertices of $\{y_1, \ldots, y_{kN}\}$ between some two vertices of X. Hence we may assume by relabeling that $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_N$ appear in that order in π, with their children somehow interspersed. We delete the remaining vertices of $y_1, \ldots, y_{kN}$ and all their children to obtain an induced subgraph G_1. Let $Y = \{y_1, \ldots, y_N\}$, and call $X \cup Y$ the *parents*. Two vertices u and v are the endpoints of two segments in π. Sometimes one of those segments does not have internal vertices from both X and Y; in this case we refer to those internal vertices as the vertices *between* u and v.

Lemma 6. *Within π, there is a subordering consisting of $X \cup Y$ and $3k$ children of each type in G_1, such that the children of any type appear consecutively.*

Proof. We iteratively select $3k$ children of some type, until we obtain all the types. Starting from a vertex a (say $a = x_1$, for example), a step ends when we reach a parent vertex or obtain $3k$ children of the same unselected type. In the latter case, select these $3k$ vertices. In either case, let the last vertex reached be a and continue.

We claim that all types are selected by the time we return to x_1. Suppose that a particular type is not selected. In each step, we see at most $3k-1$ vertices of that type. The number of steps is $r+k+N$, where r is the number of types selected. Since there are $3k(Nk+k+N)$ children of each type, we must have selected children of all Nk types. $\square$

Let G_2 be the subgraph of G_1 induced by the parents and the children selected in Lemma 6. We will show that G_2 does not embed in k pages under π. As we discard vertices to study smaller subgraphs, we refer to the ordering of the remaining vertices within π when we say that the induced subgraph has no k-page embedding under π.

We say that vertices $a_1, \ldots, a_m$ form a *twist of size* m with $b_1 \ldots, b_m$ if $a_1, \ldots, a_m, b_1, \ldots, b_m$ appear in that order in π and a_i and b_i are adjacent for $1 \le i \le m$. Note that if a vertex ordering contains a twist of size m, then every book embedding using that ordering requires at least m pages, as there are m pairwise intersecting edges induced by the vertices of the twist that require distinct pages.

A set Z of children of the same type have the same neighborhood in G. In a k-page embedding of G_2, we say that the vertices of Z have the *same edge assignment* if for every neighbor v of the vertices in Z, the edges from v to Z lie on the same page. We use $N(v)$ for the set of neighbors of vertex v in G.

Lemma 7. *In a k-page embedding of G_2 under π, the central k children of any one type have the same edge assignment.*

Proof. Let z be a child of type (i,j), and let $v_1, \ldots, v_k$ be the neighbors of z in order of their appearance in π. Group the $3k$ consecutive children of type (i,j) into three runs A, B, C of size k. For $v_r \in N(z)$, we show that all edges from v_r to B lie on the same page.

Fix vertices $a_1, \ldots, a_{r-1}$ in A and $c_{r+1}, \ldots, c_k$ in C. Given $z' \in B$, note that the vertices $a_1, \ldots, a_{r-1}, z', c_{r+1}, \ldots, c_k$ form a twist of size k with $v_1, \ldots, v_k$. Since $a_1, \ldots, a_{r-1}$ and $c_{r+1}, \ldots, c_k$ are fixed, only the edge from v_r to a vertex of B varies, and it must avoid the $k-1$ pages of the other edges in the twist. Hence all edges from v_r to B lie on the same page. Since this holds for all r, the vertices of B have the same edge assignment. $\square$

Let G_3 be the subgraph of G_2 induced by the parents and the k central children of each type. In fact, we will further restrict the vertex set by keeping only five vertices of Y and their children, along with X. The next simple observation using twists enables us to select a few special vertices of Y.

Lemma 8. *Let $x_0 = y_N$ and $x_{k+1} = y_1$. In a k-page embedding of G_3 under π, for every j with $0 \le j \le k$, at most k vertices of Y have children between x_j and x_{j+1}.*

Proof. Suppose that $\{y_{i_1}, \ldots, y_{i_{k+1}}\}$ have children between x_j and x_{j+1}, with $i_1 < \cdots < i_{k+1}$, and let z be a child of $y_{i_{j+1}}$ between x_j and x_{j+1}. Now $y_{i_1}, \ldots, y_{i_{k+1}}$ form a twist of size $k+1$ with $x_1, x_2, \ldots, x_j, z, x_{j+1}, \ldots, x_k$, preventing G_3 from embedding in k pages. $\square$

In Lemma 7, we proved that in a k-page embedding of G_3 under π, the children of any one type have the same edge assignment (and appear consecutively). By Lemma 8, at most $k(k+1)$ vertices of Y have children (in G_3) along the part of the circle from y_N to y_1 that contains X. Since $N = k^2 + k + 5 = k(k+1) + 5$, at least five vertices of Y have all their children (all k types) along the part of the circle from y_1 to y_N.

In particular, there are at least three such vertices of Y aside from y_1 and y_N. Let y_a, y_b, y_c be three such vertices, with $a < b < c$. Let $Z_{i,j}$ denote the set of k children of type (i,j) in G_3, and let $Z = \bigcup_{(i,j)\in\{a,b,c\}\times[k]} Z_{i,j}$. Let G_4 be the subgraph of G_3 induced by $X \cup \{y_1, y_a, y_b, y_c, y_N\} \cup Z$. It suffices to show that G_4 does not embed in k pages under π.

Assume henceforth that we have a k-page embedding of G_4 under π.

The sets $Z_{i,j}$ for $j \in [k]$ and $i \in \{a,b,c\}$ are located along the part of the circle from y_1 to y_N that avoids X. We say that $Z_{i,r}$ is *before* $Z_{i,s}$ if it is encountered first when following this part of the circle from y_1 to y_N (similarly define *after*).

Lemma 9. *For $r < s$, if $Z_{i,r}$ and $Z_{i,s}$ are on the same side of y_i (both before y_i or both after y_i), then $Z_{i,r}$ is before $Z_{i,s}$.*

Proof. We state the proof for when $Z_{i,r}$ and $Z_{i,s}$ are both before y_i; the other argument is symmetric. Suppose that $Z_{i,s}$ is before $Z_{i,r}$. Since $s \in [k]$, we may choose $S \subseteq Z_{i,s}$ and $R \subseteq Z_{i,r}$ with $|S| = s$ and $|R| = k+1-s$. Since the vertices of $Z_{i,j}$ are adjacent to all of $X - \{x_j\}$, we have $S \subseteq N(x_r)$ and $R \subseteq N(x_s)$. We conclude that $y_i, x_1, \ldots, x_k$ form a twist of size $k+1$ with the vertices of $S \cup R$. □

The *earlier* children of y_i are those before y_i; the others are its *later* children.

Lemma 10. *All edges joining y_i to its earlier children lie on the same page. Symmetrically, those joining y_i to its later children lie on the same page.*

Proof. Consider the earlier children of y_i. By Lemma 7, the vertices of a set $Z_{i,j}$ have the same edge assignment. Hence it suffices to show that an edge from y_i to $Z_{i,r}$ and an edge from y_i to $Z_{i,s}$ are on the same page.

We may assume that $Z_{i,r}$ is before $Z_{i,s}$. Choose $w \in Z_{i,r}$, and let z be the first vertex of $Z_{i,s}$. We have picked z so that all edges from X to the rest of $Z_{i,s}$ cross y_iz (and also y_iw). The $k-1$ vertices of $Z_{i,s} - \{z\}$ form a twist with the $k-1$ vertices of $X - \{x_s\}$. Therefore, only one page remains for y_iz and y_iw. □

Lemma 11. *If $x_1, \ldots, x_k$ form twists with both $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$, where $v_1, \ldots, v_k$ come before $w_1, \ldots, w_k$ except possibly $v_k = w_1$, then for $1 \le r \le k$ the edges incident to x_r in the two twists are on the same page.*

Proof. Observe that $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k$ form a twist with $v_1, \ldots, v_{r-1}, w_{r+1}, \ldots, w_k$. The edges x_rv_r and x_rw_r cross all $k-1$ edges formed by the twist. □

Lemma 12. *If $Z_{i,1}$ is before $Z_{i,k}$ for some i in $\{a,b,c\}$, then G_4 does not embed in k pages under π.*

Proof. The vertices of X form twists with both $\{y_1\} \cup Z_{i,1}$ and $Z_{i,k} \cup \{y_N\}$. By Lemma 11, the edges incident to x_r in the two twists are on the same page, which we call page r, for $1 \leq r \leq k$. By Lemma 7, the edges from x_r to all of $Z_{i,1} \cup Z_{i,k}$ are on the same page.

Suppose that some $Z_{i,j}$ lies after $Z_{i,1}$ and before $Z_{i,k}$. Any edge from x_r to $Z_{i,j}$ crosses the edges from $x_1, \ldots, x_{r-1}$ to $\{y_1\} \cup Z_{i,1}$ and from $x_{r+1}, \ldots, x_k$ to $Z_{i,k} \cup \{y_N\}$. Therefore, all edges from x_r to $Z_{i,j}$ lie on page r.

Since $Z_{i,1}$ is before $Z_{i,k}$, it follows that $Z_{i,1}$ is before y_i or $Z_{i,k}$ is after y_i. If both, then since $k \geq 3$, some $Z_{i,j}$ is after $Z_{i,1}$ and before $Z_{i,k}$. If $Z_{i,j}$ is before y_i, then $Z_{i,1}$ and $Z_{i,j}$ are before y_i; otherwise, $Z_{i,k}$ and $Z_{i,j}$ are after y_i. By symmetry, we may assume the former.

Let z be the first vertex of $Z_{i,j}$. Since $y_i z$ crosses the edges from $X - \{x_j\}$ to the last vertex of $Z_{i,j}$, edge $y_i z$ lies on page j. Let z' be the first vertex of $Z_{i,1}$. Since $y_i z'$ crosses the edges from $X - \{x_1\}$ to the last vertex of $Z_{i,1}$, edge $y_i z'$ lies on page 1. However, since $j \neq 1$, this contradicts Lemma 10. We conclude that G_4 does not embed in k pages under π. ☐

Lemma 13. *If $Z_{i,k}$ is before $Z_{i,1}$ for all $i \in \{a, b, c\}$, then G_4 does not embed in k pages under π.*

Proof. For $i \in \{a, b, c\}$, by Lemma 9, y_i is after $Z_{i,k}$ and before $Z_{i,1}$. Since $k \geq 3$, we may choose $j \in [k] - \{1, k\}$. Now $Z_{b,j}$ occurs before or after y_b; by symmetry, we may assume that $Z_{b,j}$ is before y_b (hence also before $Z_{b,k}$, by Lemma 9). Now consider the location of y_a.

Case 1: y_a is after some child of y_b (on the left in Fig. 1). Let $Z_{b,r}$ be the last k children of y_b before y_a. Note that $r > 1$. Now $y_b, x_1, \ldots, x_k$ form a twist of size $k+1$ with r vertices of $Z_{b,r}$, y_a, and $k - r$ vertices of $Z_{a,1}$ ($Z_{a,1}$ is after y_a by Lemma 9; this contribution is empty if $r = k$). Hence in this case G_4 does not embed in k pages under π.

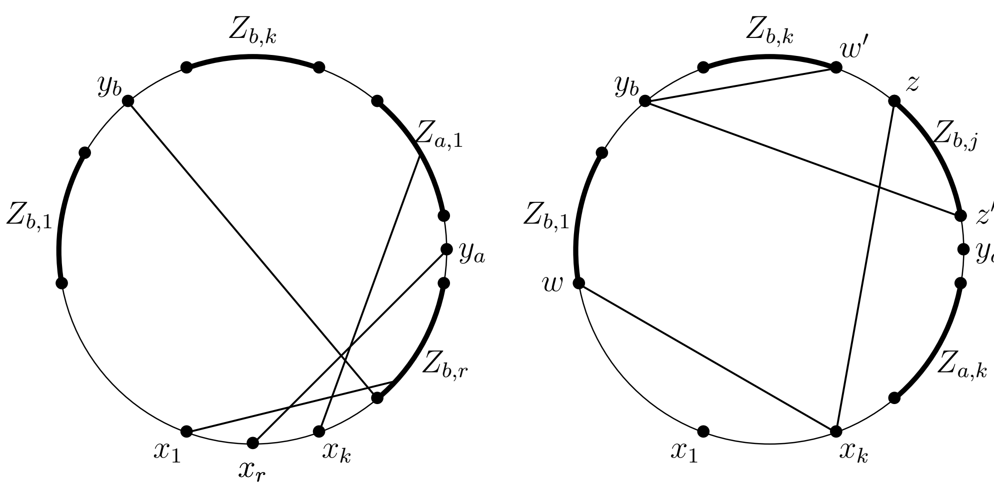

Figure 1: The cases of Lemma 13 (twist of size $k+1$, crossing on a page).

Case 2: y_a is before all children of y_b (on the right in Fig. 1). Thus y_a is before $Z_{b,j}$, and $Z_{a,k}$ is before y_a. Since $j < k$, vertices $x_1, \ldots, x_k$ form a twist with $k - 1$ vertices of $Z_{a,k}$ and the last vertex of $Z_{b,j}$ (call it z). Also recall that $x_1, \dots, x_k$ form a twist with $\{y_b\} \cup Z_{b,1}$. By Lemma 11, $x_k z$ and $x_k w$ lie on the same page, where w is the last vertex of $Z_{b,1}$.

Let w' be the first vertex of $Z_{b,k}$. Note that $x_1, \dots, x_k$ form a twist with $(Z_{b,k} - \{w'\}) \cup \{w\}$. Since $y_b w'$ crosses its $k-1$ edges other than $x_k w$, edges $y_b w'$ and $x_k w$ lie on the same page.

Finally, by Lemma 10, $y_b w'$ lies on the same page with $y_b z'$, where z' is the first vertex of $Z_{b,j}$. Now $y_b z'$ and $x_k z$ lie on the same page, but they cross. Hence in this case also G_4 does not embed in k pages under π. □

Lemmas 12 and 13 eliminate all possibilities for k-page embeddings and complete the proof of the theorem.

Finally, we remark that the k-tree G constructed for the proof of Theorem 1 has a smooth tree-decomposition with a host tree of maximum degree $k+2$. Let $X_i = X \cup \{y_i\}$ for $1 \le i \le kN$. Form a path with vertices $X_1, \dots, X_{kN}$. For each X_i and x_j, form a path with endpoint X_i whose vertices correspond to bags formed by adding to $X_i - \{x_j\}$ one child of type (i, j). This is the desired tree-decomposition of G. As mentioned in the introduction, this leaves the question of what is the largest degree of host trees in tree-decompositions of k-trees that guarantees the existence of a k-page embedding.

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