Math 241 C1H - More Applications of Multiple Integrals

Example I: Computing the mass of something with varying density.

Suppose you have a solid cone and you want to determine its mass. However, the material at the tip of the cone is twice as dense as the material at the base. Say that the density of the material at the base is ρ, the cone has radius r_0 and height h. What is the total mass?

Let's say the base of the cone is in the plane $z = 0$ and the tip of the cone is at $z = h$. Also, assume the density $\rho(x, y, z)$ of the cone varies linearly with z. Then, we have $\rho(x, y, 0) = \rho$ and $\rho(x, y, h) = 2\rho$. Thus,

$$\rho(x, y, z) = \rho\left(1 + \frac{z}{h}\right).$$

Then, the total mass is

$$M = \iiint_W \rho(x, y, z)\, dV.$$

We switch to cylindrical coordinates. We integrate from $0 \le z \le h$, $0 \le \theta \le 2\pi$, and $0 \le r \le r_0 \frac{h-z}{h}$. This gives

$$\begin{aligned}
M &= \int_0^h \int_0^{2\pi} \int_0^{r_0(h-z)/h} \rho\left(1 + \frac{z}{h}\right) r\, dr\, d\theta\, dz \\
&= \int_0^h \int_0^{2\pi} \rho r_0^2/2 \left(\frac{h-z}{h}\right)^2 \left(\frac{h+z}{h}\right) d\theta\, dz \\
&= \frac{\pi\rho r_0^2}{h^3} \int_0^h (h^2 - z^2)(h - z)\, dz \\
&= \frac{\pi\rho r_0^2}{h^3} \int_0^h \left(h^3 - hz^2 - zh^2 + z^3\right) dz \\
&= \frac{\pi\rho r_0^2}{h^3} \left[h^3 z - hz^3/3 - z^2h^2/2 + z^4/4\right]_0^h \\
&= \frac{\pi\rho r_0^2}{h^3} \left[h^4 - h^4/3 - h^4/2 + h^4/4\right] \\
&= \frac{5\pi r_0^2 h}{12} \\
&= \frac{\pi r_0^2 h}{3} \cdot \frac{5\rho}{4}.
\end{aligned}$$

This shows that the average density of the cone is $5\rho/4$.

Example II: Average outcome.

Pick two points at random in the rectangle $[0, 1] \times [0, 1]$, say (x_1, y_1) and (x_2, y_2). What is the average length of the line segment between them?

The length of the line segment between them is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

To compute the average length, we must average over all choices of x_1, x_2, y_1 and y_2. This gives us the integral

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\, dx_1\, dy_1\, dx_2\, dy_2.$$

It is possible to show that the integral above is equal to

$$\frac{1}{15}\left[2 + \sqrt{2} + 5\ln(1 + \sqrt{2})\right] \approx 0.5214...$$

Hence, the average length of the line segment is just over 1/2.