Examples of L-functions:

1. Dirichlet L-functions

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

These have an analytic continuation to all of $\mathbb{C}$ and a functional equation of the usual type.

2. L-functions associated to modular forms.

As we proved, these have an analytic continuation to $\mathbb{C}$ and a functional equation as well.

3. Dedekind ζ-functions. If $K/\mathbb{Q}$ is a number field, let

$$\zeta_K(s) = \sum_{\substack{\mathfrak{a} \subseteq O_K \\ \mathfrak{a} \neq 0}} \frac{1}{N(\mathfrak{a})^s}.$$

As is (or should be proved) in Math 530, these have an meromorphic continuation to $\mathbb{C}$ and a functional equation of the usual type. (They have a simple pole at $s = 1$ just like $\zeta(s)$).

4. Artin L-functions

Suppose that $K/\mathbb{Q}$ is a Galois extension, and $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_d(\mathbb{C})$ is a homomorphism. If p is unramified in K, the Euler factor at the prime p is given by

$$\det(1 - \rho(\mathrm{Frob}_p)p^{-s})^{-1}.$$

This is the inverse of the characteristic polynomial of $\rho(\mathrm{Frob}_p)$ evaluated at p^{-s}. Let

$$L(\rho, s) = \prod_p \det(1 - \rho(\mathrm{Frob}_p)p^{-s})^{-1}.$$

(There are Euler factors at the primes that ramify, but the explicit recipe for what they are is somewhat complicated).

In the case when ρ is irreducible, these are conjectured to be entire, but this is not known. (It is known that they have a meromorphic continuation, and satisfy a functional equation of the usual sort).

General definition of an L-function. (Iwaniec and Kowalski, pg. 94).

5. Rankin-Selberg L-functions

Suppose for simplicity that

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(1))$$

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_\ell(\Gamma_0(1))$$

are normalized Hecke eigenforms. Then,

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^{s+\frac{k-1}{2}}} \\ &= \prod_p (1-\alpha_p(f)p^{-s})^{-1}(1-\beta_p(f)p^{-s})^{-1} \\ L(g,s) &= \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+\frac{k-1}{2}}} \\ &= \prod_p (1-\alpha_p(g)p^{-s})^{-1}(1-\beta_p(g)p^{-s})^{-1}. \end{aligned}$$

We define

$$L(f\otimes g,s) = \prod_p (1-\alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1-\alpha_p(f)\beta_p(g)p^{-s})^{-1}$$
$$(1-\beta_p(f)\alpha_p(g)p^{-s})^{-1}(1-\beta_p(f)\beta_p(g)p^{-s})^{-1}.$$

Fact: We have

$$L(f\otimes g,s) = \zeta(2s)\sum_{n=1}^{\infty} \frac{a(n)b(n)}{n^{s+\frac{k+\ell}{2}-1}}.$$

Proof: Clearly, $a(n)b(n)$ is multiplicative. It suffices to prove the desired statement therefore when n is a prime power. Fix a prime p, and let $a = \alpha_p(f)$, $b = \beta_p(f)$, $c = \alpha_p(g)$ and $d = \beta_p(g)$. Then, the Dirichlet coefficients at powers of p are given by

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{a(p^r)}{p^{r(k-1)/2}} z^r &= \frac{1}{1-az}\frac{1}{1-bz} \\ &= \sum_{r=0}^{\infty} (a^r + a^{r-1}b + \cdots + ab^{r-1} + b^r) z^r \\ &= \sum_{r=0}^{\infty} \frac{a^{r+1}-b^{r+1}}{a-b} z^r. \end{aligned}$$

Similarly,

$$\sum_{r=0}^{\infty} \frac{b(p^r)}{p^{r(\ell-1)/2}} = \sum_{r=0}^{\infty} \frac{c^{r+1}-d^{r+1}}{c-d} z^r.$$

Thus, we have

$$\sum_{r=0}^{\infty}\frac{a(p^r)b(p^r)}{p^{r\left(\frac{k+\ell}{2}-1\right)}}z^r = \sum_{r=0}^{\infty}\frac{1}{(a-b)(c-d)}(a^{r+1}-b^{r+1})(c^{r+1}-d^{r+1})z^r$$

$$= \frac{1}{(a-b)(c-d)}\sum_{r=0}^{\infty}((ac)^{r+1}-(ad)^{r+1}-(bc)^{r+1}+(bd)^{r+1})z^r$$

$$= \frac{1}{(a-b)(c-d)}\left(\frac{ac}{1-acz}-\frac{ad}{1-adz}-\frac{bc}{1-bcz}+\frac{bd}{1-bdz}\right)$$

$$= \frac{1}{(a-b)(c-d)}\left(\frac{ac(1-adz)(1-bcz)(1-bdz)}{(1-acz)(1-adz)(1-bcz)(1-bdz)}\right.$$

$$-\frac{ad(1-acz)(1-bcz)(1-bdz)}{(1-acz)(1-adz)(1-bcz)(1-bdz)}$$

$$-\frac{bc(1-acz)(1-adz)(1-bdz)}{(1-acz)(1-adz)(1-bcz)(1-bdz)}$$

$$\left.+\frac{bd(1-acz)(1-adz)(1-bcz)}{(1-acz)(1-adz)(1-bcz)(1-bdz)}\right)$$

Multiplying out the numerator, we see that it is

$$(a-b)(c-d)(1-abcdz^2).$$

Note that $ab = 1$, $cd = 1$ and so $abcd = 1$. Thus, we have

$$\sum_{r=0}^{\infty}\frac{a(p^r)b(p^r)}{p^{r+\frac{k+\ell}{2}-1}}z^r = \frac{(1-z^2)}{(1-acz)(1-adz)(1-bcz)(1-bdz)}.$$

Setting $z = p^{-s}$ and multiplying over all p, we get

$$\sum_{n=1}^{\infty}\frac{a(n)b(n)}{n^{s+\frac{k+\ell}{2}-1}} = \prod_p (1-p^{-2s})(1-\alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1-\alpha_p(f)\beta_p(g)p^{-s})^{-1}$$

$$(1-\beta_p(f)\alpha_p(g)p^{-s})^{-1}(1-\beta_p(f)\beta_p(g)p^{-s})^{-1}$$

$$= \frac{1}{\zeta(2s)}L(f\otimes g, s).$$