

Math 597 - Homework 1

1. Show that if $G = \mathrm{PSL}_2(\mathbb{Z})$, and $H \subseteq G$ is a finite group, then $|H| \leq 6$.
 - (a) Suppose that $M \in \mathrm{SL}_2(\mathbb{Z})$ has finite order. Show that M is diagonalizable over \mathbb{C} and that $M = \pm I$ or $|\mathrm{tr} M| < 2$.
 - (b) Let $K = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv d \equiv 1 \pmod{2}, b \equiv c \equiv 0 \pmod{2} \right\}$. Show that $K \trianglelefteq G$, $|G : K| = 6$, and that the only elements in K of finite order are $\pm I$.
 - (c) Let H be a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $-I$. Using that $HK/K \cong H/(H \cap K)$, conclude the desired result.
2. (a) Show that if $f(z)$ is a modular function of weight 0, then $\frac{d}{dz}f(z)$ is a meromorphic modular form of weight 2.
 - (b) Let $f(z)$ be a weakly holomorphic modular form of weight 2 with Fourier expansion

$$\sum_n a(n)q^n.$$

Show that $a(0) = 0$.

3. Evaluate

$$\sum_{n=1}^{\infty} \sigma(n)e^{-2\pi n}.$$

4. Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ is a nonzero cusp form with real coefficients. Prove that infinitely many of the $a(n)$ are negative.
5. Prove that $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.
6. This exercise will reprove the fact that if

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k,$$

then so is

$$\sum_{n=0}^{\infty} (a(pn) + p^{k-1}a(n/p))q^n,$$

(where we define $a(k) = 0$ if k is not an integer).

- (a) For a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ and a function $f : \mathbb{H} \rightarrow \mathbb{C}$, define

$$f|_k M = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right)$$

where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\Gamma_0(p) = \left[\begin{array}{l} a \quad b \\ c \quad d \end{array} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{p} \right]$. Show that if $f|_k M = f$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$ then if $g(z) = f(pz)$, then $g|_k M = g$ for all $M \in \Gamma_0(p)$.

(b) Define $\mathrm{Tr}_p(f) = \sum_{M \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_0(p)} f|M$. Show that $\mathrm{Tr}_p(f)|_k M = \mathrm{Tr}_p(f)$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$.

(c) Compute an explicit set of representatives for the right cosets of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{Z})$ and show that

$$p^{k-1} \mathrm{Tr}_p(f) = \sum_{n=0}^{\infty} (a(pn) + p^{k-1} a(n/p)) q^n.$$

7. Prove that if p is prime, then $\tau(p)$ is even. Use this to determine necessary and sufficient conditions on when $\tau(n)$ is even.