Math 597 - Homework 3

1. Let

$$f(z) = \eta(z)^2\eta(11z)^2 = \sum_{n=1}^{\infty} c(n)q^n.$$

(a) Verify numerically that for $p \leq 100$, $p \neq 2, 11$,

$$c(p) = p + 1 - \#E(\mathbb{F}_p),$$

where $E : y^2 + y = x^3 - x^2 - 10x - 20$. (Hereafter you may assume that $c(p) = p+1-\#E(\mathbb{F}_p)$).

(b) Compute the density of primes p for which $c(p)$ is odd.

(c) Compute the density of primes p for which $c(p) \equiv 0 \pmod{11}$.

(d) Show that $c(p) \equiv \tau(p) \pmod{11}$.

2. Let $E_1 : y^2 = x^3 - 27x - 42$ and $E_2 : y^2 + y = x^3 - x$. Let $a_1(p) = p + 1 - \#E_1(\mathbb{F}_p)$ and $a_2(p) = p + 1 - \#E_2(\mathbb{F}_p)$.

(a) Numerically estimate the density of primes p for which $3|a_1(p)$. Do the same for $a_2(p)$. What do you observe?

(b) Numerically estimate the density of primes p for which $9|a_1(p)$. Do the same for $a_2(p)$. What do you observe? How can you explain this?

3. Suppose that $f = \sum_{n=1}^{\infty} a(n)q^n$ is a cusp form on the full modular group and that $a(n) \in \mathbb{Z}$ for all n. Suppose there are distinct primes t and ℓ so that $t|a(n)$ whenver $\ell|n$. Show that $t|a(n)$ for all n.

[Hint: Write $f = \sum c_i f_i$ as a linear combination of Hecke eigenfroms. The c_i and the coefficients of the f_i will lie in some number field L. Fix a prime ideal $\mathfrak{l}$ above ℓ and look at the mod $\mathfrak{l}^k$ Galois representations associated to the f_i for a high enough power of $\mathfrak{l}^k$ that you can read off mod $\mathfrak{l}$ information about the coefficients of f].