Math241, Quiz 5-version a, Oct 15

Name: Solutions.

Question 1: [5pt] Find the maximum and minimum values of the function $f(x,y,z)$ subject to the given constraints. Hint:use Lagrange multipliers.

$$f(x,y,z) = x^2+y^2-z^2; \quad x^4+y^4+z^4=1$$

Sol) $\nabla f = \lambda \nabla g$, $g(x,y,z) = x^4+y^4+z^4$,

$\langle 2x, 2y, -2z\rangle = \lambda \langle 4x^3, 4y^3, 4z^3\rangle$

$$\begin{cases} 2x = 4x^3\lambda, & 2x(1-2x^2\lambda)=0, & x=0 \text{ or } x^2=\frac{1}{2\lambda}, \\ 2y = 4y^3\lambda & 2y(1-2y^2\lambda)=0, & y=0 \text{ or } y^2=\frac{1}{2\lambda}, \\ -2z = 4z^3\lambda & 2z(1+2z^2\lambda)=0, & z=0 \text{ or } z^2=-\frac{1}{2\lambda}, \\ x^4+y^4+z^4=1 \end{cases}$$

Note that $\lambda \neq 0$ if so $(x,y,z)=(0,0,0)$ but $(0,0,0)$ does not satisfy $x^4+y^4+z^4=1$.

① $x=y=0$, $z^2=-\frac{1}{2\lambda}$, $(0,0,\pm 1)$, $f(0,0,\pm1) = -1$, min.val.

② $x=z=0$, $y^2=\frac{1}{2\lambda}$, $(0,\pm1,0)$, $f(0,\pm1,0)=1$,

③ $y=z=0$, $x^2=\frac{1}{2\lambda}$, $(\pm1,0,0)$, $f(\pm1,0,0)=1$,

④ $z=0$, $x^2=y^2=\frac{1}{2\lambda}$, $\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1$, $\frac{1}{2\lambda^2}=1$, $\lambda^2=\frac{1}{2}$, $\lambda=\frac{1}{\sqrt{2}}$.

$\left(\pm\sqrt{\frac{\sqrt{2}}{2}}, \pm\sqrt{\frac{\sqrt{2}}{2}}, 0\right)$, $f\left(\pm\sqrt{\frac{\sqrt{2}}{2}}, \pm\sqrt{\frac{\sqrt{2}}{2}}, 0\right) = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} = \sqrt{2}$ max. val.

Note that $x^2=\frac{1}{2\lambda}$, $z^2=-\frac{1}{2\lambda}$ does not occur at the same time.

Question 2: [5pt] Find the volume of the solid under the hyperbolic paraboloid $z = 16 + x^2 - y^2$ and above the square region $R = [0,2] \times [1,4]$.

Sol) $\displaystyle\int_0^2 \int_1^4 16 + x^2 - y^2 \, dy \, dx$

$$= \int_0^2 \left[16y + x^2 y - \frac{y^3}{3} \right]_1^4 dx = \int_0^2 64 + 4x^2 - \frac{64}{3} - \left(16 + x^2 - \frac{1}{3}\right) dx$$

$$= \int_0^2 \underbrace{48 - \frac{64}{3} + \frac{1}{3}}_{} + 3x^2 \, dx = \int_0^2 27 + 3x^2 \, dx =$$

$$= 48 - \frac{63}{3} = 48 - 21 = 27$$

$$\left[27x + x^3 \right]_0^2 = 54 + 8 = 62.$$