Math241, Quiz 7-version b, Nov. 5

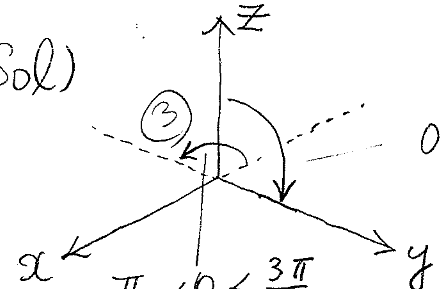
Name: Solutions

Question 1: [5pt] Evaluate the following integral.

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV,$$

where E is bounded by the sphere $x^2+y^2+z^2=9$ in the third octant.

Sol)



$0 \le \phi \le \frac{\pi}{2}$, $\quad (\rho, \theta, \phi)$

$\rho^2 = x^2+y^2+z^2$

$\pi \le \theta \le \frac{3\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\int_\pi^{\frac{3\pi}{2}}\int_0^3 e^\rho \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \int_0^{\frac{\pi}{2}} \sin\phi \int_\pi^{\frac{3\pi}{2}}\int_0^3 \rho^2 e^\rho \, d\rho\, d\theta\, d\phi$$

$$= \left(\int_\pi^{\frac{3\pi}{2}}\int_0^3 \rho^2 e^\rho \, d\rho\, d\theta\right)\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi = \frac{\pi}{2}\underbrace{\int_0^3 \rho^2 e^\rho \, d\rho}_{(*)}\ \underbrace{\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi}_{[-\cos\phi]_0^{\frac{\pi}{2}} = 1} = \boxed{\frac{\pi}{2}(5e^3-2)}$$

(*) $\left\{ \begin{array}{l} u=\rho^2, \ dv = e^\rho d\rho \\ du = 2\rho d\rho, \ v = e^\rho \end{array} \right\}$

$$[\rho^2 e^\rho]_0^3 - 2\underbrace{\int_0^3 \rho e^\rho d\rho} = 9e^3 - 2(2e^3+1) = 5e^3-2$$

$\left\{ \begin{array}{l} u=\rho, \ dv = e^\rho d\rho \\ du = d\rho, \ v = e^\rho \end{array} \right\}$

$$[\rho e^\rho]_0^3 - \int_0^3 e^\rho d\rho = 3e^3 - [e^\rho]_0^3 = 3e^3 - (e^3-1) = 2e^3+1$$

Question 2: [5pt] Use the given ~~translation~~ transformation to evaluate the integral.

$$\iint_R (5x - 3y)\,dA,$$

where R is the parallelogram with vertices (3,-1), (1,-7), (-3,1), and (-1,7) in the xy-plane;

$$x = u + 2v, \quad y = 3u - 4v.$$

Sol) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} = -4-6 = -10.$

For the integral domain S in the uv-plane,

$u = x - 2v, \quad y = 3(x-2v) - 4v, \quad y = 3x - 10v,$

$v = \frac{1}{10}(3x - y), \quad u = x - 2v = x - \frac{1}{5}(3x-y) = \frac{2}{5}x + \frac{1}{5}y$

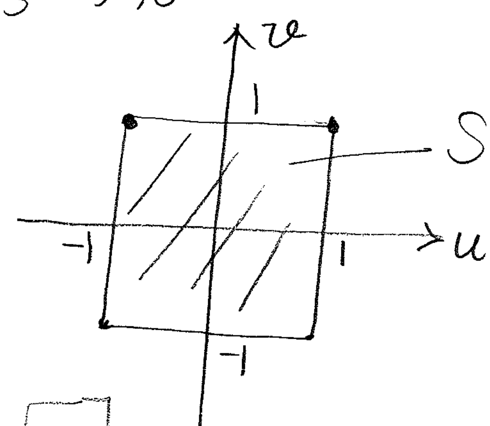
$$\begin{cases} u = \frac{1}{5}(2x+y) \\ v = \frac{1}{10}(3x-y) \end{cases}$$

$(3,-1) \to (\frac{1}{5}\cdot 5, \frac{1}{10}\cdot 10) = (1,1),$

$(1,-7) \to (\frac{1}{5}(-5), \frac{1}{10}\cdot 10) = (-1,1),$

$(-3,1) \to (\frac{1}{5}(-5), \frac{1}{10}\cdot(-10)) = (-1,-1),$

$(-1,7) \to (\frac{1}{5}\cdot 5, \frac{1}{10}(-10)) = (1,-1).$



$\int_{-1}^{1}\int_{-1}^{1}\big(5\cdot(u+2v) - 3(3u-4v)\big)\cdot 10\,du\,dv =$

$10\int_{-1}^{1}\int_{-1}^{1}(\underbrace{-4u}_{\text{odd},\ -1\le u\le 1} + 22v)\,du\,dv =$

$10\int_{-1}^{1} 22\,[vu]_{-1}^{1}\,dv = 220\int_{-1}^{1}\underbrace{2v}_{\text{odd},\ -1\le v\le 1}\,dv = \boxed{0}.$