Steady state temperature; Laplacian; Dirichlet problems.

Suppose we have an insulated wire, a plate, or a 3-dimensional object. We apply certain fixed temperatures on the ends of the wire, the edges of the plate or on all sides of the 3-dimensional object. We wish to find out what is the steady state temperature distribution. That is, we wish to know what will be the temperature after long enough period of time.

We are really looking for a solution to the heat equation that is not dependent on time. Let us first do this in one space variable. We are looking for a function u that satisfies

$$u_t = k u_{xx}, \tag{1}$$

but such that $u_t = 0$ for all x and t. Hence, we are looking for a function of x alone that satisfies $u_{xx} = 0$. It is easy to solve this equation by integration and we see that $u = Ax + B$ for some constants A and B.

Suppose we have an isolated wire, and we apply constant temperature T_1 at one end (say where $x = 0$) and T_2 and the other end (at $x = L$ where L is the length of the rod). Then our steady state solution is

$$u(x) = \frac{T_2 - T_1}{L} x + T_1. \tag{2}$$

This agrees with our common sense intuition with how the heat should be distributed in the wire. So in one dimension, the steady state solutions are basically just straight lines.

Things are much more complicated in two or more dimensions. Let us restrict to two dimensions for simplicity. The heat equation in two variables is

$$u_t = k(u_{xx} + u_{yy}), \tag{3}$$

or more commonly written as $u_t = k\Delta u$ or $u_t = k\nabla^2 u$. Here the Δ and ∇^2 symbols mean $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. I will use Δ from now on. The reason for that notation is that you can define Δ to be the right thing for any number of space dimensions and then the heat equation is always $u_t = k\Delta u$. The Δ is called the *Laplacian*.

OK, now that we have notation out of the way let us see what does an equation for the steady state solution look like. We are looking for a solution to (3) which does not depend on t. Hence we are looking for a function $u(x, y)$ such that

$$\Delta u = u_{xx} + u_{yy} = 0. \tag{4}$$

This is called the *Laplace equation*. Solutions to this equation are called *harmonic functions* and have many nice properties and applications far beyond the steady state heat solution.

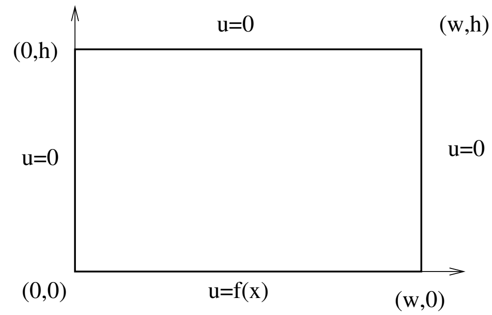
Harmonic functions in two variables are no longer just linear (plane graphs). For example, you can check that the functions $x^2 - y^2$ and xy are harmonic. However, if you remember your multivariable calculus we note that if u_{xx} is positive, u is concave up in the x direction, then u_{yy} must be negative and u must be concave down in the y direction. Therefore a harmonic function can never have any "hilltop" or "valley" on the graph. This observation is consistent with our intuitive idea of steady state heat distribution.

Usually the Laplace equation is part of a so-called *Dirichlet problem*. That is, we have some region in the xy-plane and we specify certain values along the boundaries of the region. We then try to find a solution u defined on this region such that u agrees with the values we specified on the boundary.

For simplicity we will consider a rectangular region. Also for simplicity we will specify boundary values to be zero at 3 of the four edges and only specify an arbitrary function at one edge. As we still have the principle of superposition, you can use this solution to derive the general solution for arbitrary boundary values by solving 4 different sets of equations, one for each edge, and adding the solutions together.

So suppose that we wish to solve the following problem. Let h and w be the height and width of our rectangle, with one corner at the origin and lying in the first quadrant.

$$
\begin{aligned}
&\Delta u = 0, && (5)\\
&u(0, y) = 0 \text{ for } 0 < y < h, && (6)\\
&u(x, h) = 0 \text{ for } 0 < x < w, && (7)\\
&u(w, y) = 0 \text{ for } 0 < y < h, && (8)\\
&u(x, 0) = f(x) \text{ for } 0 < x < w. && (9)
\end{aligned}
$$



The method we will apply is separation of variables, to come up with enough solutions such that we can use the Fourier series for $f(x)$ to solve the problem. Hence we try $u(x, y) = X(x)Y(y)$. We plug into the equation to get

$$X''Y + XY'' = 0. \tag{10}$$

We put the Xs on one side and the Ys on the other to get

$$-\frac{X''}{X} = \frac{Y''}{Y}. \tag{11}$$

The left hand side only depends on x and the right hand side only depends on y. This means that there is some constant λ such that $\lambda = \frac{-X''}{X} = \frac{Y''}{Y}$. And we get two equations

$$X'' + \lambda X = 0, \tag{12}$$

$$Y'' - \lambda Y = 0. \tag{13}$$

Furthermore from the boundary conditions we get that $X(0) = X(w) = 0$ and $Y(h) = 0$. Taking the equation for X we have already seen that we have a nontrivial solution if and only if $\lambda = \lambda_n = \frac{n^2\pi^2}{w^2}$ and the solution is a multiple of

$$X_n(x) = \sin \frac{n\pi}{w} x. \tag{14}$$

This means that the general solution for Y is

$$Y_n(y) = A_n \cosh \frac{n\pi}{w} y + B_n \sinh \frac{n\pi}{w} y. \tag{15}$$

We only have one condition on Y_n and hence we can just pick one of the variables to be whatever is convenient. It will be useful to have $Y_n(0) = 1$, so we could pick $A_n = 1$. Setting $Y_n(h) = 0$ and solving for B_n we get that

$$B_n = \frac{-\cosh \frac{n\pi h}{w}}{\sinh \frac{n\pi h}{w}}. \tag{16}$$

After a little bit of simplification we find

$$Y_n(y) = \frac{\sinh \frac{n\pi(h-y)}{w}}{\sinh \frac{n\pi h}{w}}. \tag{17}$$

We define $u_n(x, y) = X_n(x)Y_n(y)$. And note that u_n satisfies (5)–(8).

Observe that

$$u_n(x, 0) = X_n(x)Y_n(0) = \sin \frac{n\pi}{n} x. \tag{18}$$

Suppose that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{w}. \tag{19}$$

Then we get a solution of (5)–(9) of the following form.

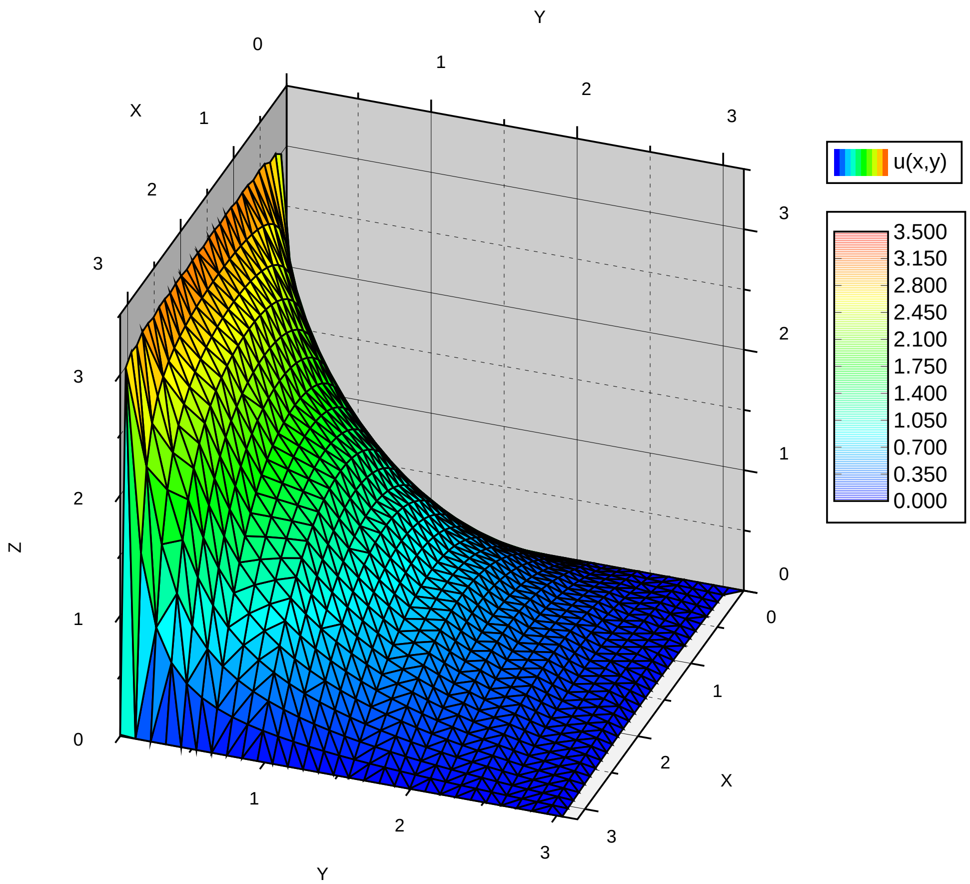
$$u(x,y) = \sum_{n=1}^{\infty} b_n u_n(x,y) = \sum_{n=1}^{\infty} b_n \left(\frac{\sinh \frac{n\pi(h-y)}{w}}{\sinh \frac{n\pi h}{w}} \right) \left(\sin \frac{n\pi}{w} x \right). \tag{20}$$

As u_n satisfies (5)–(8) and any linear combination (finite or infinite) of u_n must also satisfy (5)–(8), we see that u must satisfy (5)–(8). By plugging in $y = 0$ it is easy to see that u satisfies (9) as well.

For example, suppose that we take $w = h = \pi$ and we let $f(x) = \pi$. We find that for $0 < x < \pi$ we have

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n} \sin nx. \tag{21}$$

We then get the following graph of the temperature. This scenario corresponds to the steady state temperature on a square plate of width π with 3 sides held at 0 degrees and one side held at π degrees.



If you have arbitrary initial data on all sides then solve four problems, each using one piece of nonhomogenous data. Then use the principle of superposition to add up all four solutions to have a solution to your original problem.