SEC. 9.3–9.4 OVERVIEW — SOLVING ODES WITH FOURIER SERIES

[Read this in conjunction with the examples in class and homework.]

Section 9.3

We solve the *boundary value* problem

$$\boxed{x''(t) + ax(t) = f(t), \qquad 0 < t < L}$$

under either

- *Dirichlet* boundary conditions $x(0) = x(L) = 0$: write both $f(t)$ and $x(t)$ as Fourier *sine* series,
- *Neumann* boundary conditions $x'(0) = x'(L) = 0$: write both $f(t)$ and $x(t)$ as Fourier *cosine* series,

then substitute into the DE and equate coefficients of sines/cosines to find the coefficients of $x(t)$ in terms of the coefficients of $f(t)$.

Section 9.4

We seek the general solution $x = x_c + x_p$ of the *mechanical vibration* problem

$$\boxed{mx''(t) + cx'(t) + kx(t) = f(t)}$$

assuming that the forcing function $f(t)$ is $2L$-**periodic**.

1. We know the complementary solution from Section 3.4. For example, in the undamped case ($c = 0$):

$$x_c(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

2. To find the particular solution $x_p(t)$: if $f(t)$ is just a sine or cosine term then we know how to solve by Undetermined Coefficients (see Section 3.6).

If $f(t)$ is a sum of finitely many sine and cosine terms then we just find the particular solution corresponding to each one, and add up these particular solutions to get x_p.

In the general case where $f(t)$ is a periodic function, such as a square wave, we first decompose $f(t)$ as an infinite sum of sines and cosines (by writing f as a Fourier series) and then find the particular solution corresponding to each one, finally adding up these particular solutions to get x_p. That is, we write

$$f(t) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty}[c_n \cos\frac{n\pi t}{L} + d_n \sin\frac{n\pi t}{L}]$$

as a full Fourier series and guess

$$x_p(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}],$$

then substitute into the DE to get formulas for a_0, a_n, b_n in terms of c_0, c_n, d_n.

WARNING: If $c = 0$ (no damping) and $\omega_0 = N\pi/L$ for some $N \geq 1$, then resonance occurs at the N-th forcing frequency. This means we must multiply our guess by t at the N-th frequency level, so that we guess

$$x_p(t) = \frac{1}{2}a_0 + t[a_N \cos\frac{N\pi t}{L} + b_N \sin\frac{N\pi t}{L}] + \sum_{\substack{n=1 \\ n\neq N}}^{\infty}[a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}].$$