Here is the problem I did not do quite right in class. Turns out I should have just kept going and I would have been fine.

The question is: Find the steady periodic solution (a particular solution x_p with same period as F) to the equation

$$2x'' + 18\pi^2 x = F(t)$$

where

$$F(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ -1 & \text{if } -1 < t < 0 \end{cases},$$

extended periodically. We noted that

$$F(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t).$$

The solution must look like

$$x(t) = c_1 \cos(3\pi t) + c_2 \sin(3\pi t) + x_p(t)$$

We note that if we just tried a fourier series with $\sin(n\pi t)$ as usual, we would get duplication when $n = 3$. Therefore, we pull out that term and multiply by t. And we have add a cosine term to get everything right. That is we must try (see the fourier series DE overview):

We try

$$x_p(t) = a_3 t \cos(3\pi t) + b_3 t \sin(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} b_n \sin(n\pi t).$$

Now compute:

$$x_p''(t) = -6a_3\pi \sin(3\pi t) - 9\pi^2 a_3 t \cos(3\pi t) + 6b_3\pi \cos(3\pi t) - 9\pi^2 b_3 t \sin(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} -n^2\pi^2 b_n \sin(n\pi t).$$

Hence

$$\begin{aligned} 2x_p'' + 18\pi^2 x = {} & -12a_3\pi \sin(3\pi t) - 18\pi^2 a_3 t \cos(3\pi t) + 12b_3\pi \cos(3\pi t) - 18\pi^2 b_3 t \sin(3\pi t) + \\ & + 18\pi^2 a_3 t \cos(3\pi t) \qquad + 18\pi^2 b_3 t \sin(3\pi t) + \\ & + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-2n^2\pi^2 b_n + 18\pi^2 b_n) \sin(n\pi t). \end{aligned}$$

If we simplify:

$$2x_p'' + 18\pi^2 x = -12a_3\pi \sin(3\pi t) + 12b_3\pi \cos(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-2n^2\pi^2 b_n + 18\pi^2 b_n) \sin(n\pi t).$$

Now this series has to equal to the series for $F(t)$. So equate the coefficients and solve for a_3 and b_n.

$$\begin{aligned} a_3 &= \frac{4/(3\pi)}{-12\pi} = \frac{-1}{9\pi^2} \\ b_3 &= 0 \\ b_n &= \frac{4}{n\pi(18\pi^2 - 2n^2\pi^2)} = \frac{2}{\pi^3 n(9 - n^2)} \quad \text{for } n \text{ odd and } n \neq 3. \end{aligned}$$

That is:

$$x_p(t) = \frac{-1}{9\pi^2} t \cos(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} \frac{2}{\pi^3 n(9 - n^2)} \sin(n\pi t).$$