3.2 (SUMMARY) — GENERAL SOLUTIONS OF LINEAR EQUATIONS

[Compare all this material with the second order case, in Section 3.1.]

Consider the n-th order linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \tag{1}$$

where $p_0, \dots, p_{n-1}$ and f are continuous functions. *Example.* $y''' + x^4y'' - 3y' + y = \sin(x)$.

Theorem 2. *(Existence and uniqueness)* For each choice of numbers $a, b_0, b_1, \dots, b_{n-1} \in \mathbb{R}$, there exists exactly one solution of (1) that satisfies the initial conditions

$$y(a) = b_0,\ y'(a) = b_1,\ \dots,\ y^{(n-1)}(a) = b_{n-1}.$$

Proof: omitted.

Remark. To find this solution, we first have to treat the homogeneous ($f = 0$) equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{2}$$

Theorem 1. *(Superposition)* If $y_1, \dots, y_n$ are solutions of (2) and $c_1, \dots, c_n \in \mathbb{R}$ then the linear combination

$$y = c_1y_1 + \cdots + c_ny_n \tag{3}$$

is also a solution of (2). *Proof:* exercise. *Note.* Superposition is valid for (2), not for (1).

Example. If y_1 and y_2 are solutions of (2), then so is $5y_1 - 3y_2$.

Theorem 4. *(General solution)* Every solution of (2) can be written as a linear combination (3) of the solutions $y_1, \dots, y_n$, provided that $y_1, \dots, y_n$ are linearly independent (which means that no one of them is equal to a linear combination of the others). *Proof:* omitted.

Thus we can call $y = c_1y_1 + \cdots + c_ny_n$ the general solution of (2).

[Note that when $n = 2$, the new definition of linear independence for y_1, y_2 is the same as the old one, which says that neither y_1 nor y_2 is a multiple of the other.]

Conclusion. We should first try to find n linearly independent solutions of the homogeneous equation (2), and then we should apply the initial conditions to $y = c_1y_1 + \cdots + c_ny_n$ in order to solve for $c_1, \dots, c_n$.

Question. For constant coefficient homogeneous equations, the methods of Section 3.3 tell us how to find n solutions $y_1, \dots, y_n$. But are these solutions linearly independent. . . ?

Questions (These are examinable.)

1. Suppose $r_1 < r_2 < r_3$. Show that the functions $y_1 = e^{r_1x}, y_2 = e^{r_2x}, y_3 = e^{r_3x}$ are linearly independent.
2. Suppose $r_1 < r_2 < \ldots < r_n$. Show that the functions $y_1 = e^{r_1x}, y_2 = e^{r_2x}, \dots, y_n = e^{r_nx}$ are linearly independent.
3. Let $r \in \mathbb{R}$. Show that $y_1 = e^{rx}, y_2 = xe^{rx}, y_3 = x^2e^{rx}$ are linearly independent.
4. Show that the functions $y_1 = \sin x, y_2 = \sin(2x)$ are linearly independent.

Answer. By arguments like above, one can show that the solutions of a constant coefficient homogeneous linear equation found by the methods of Section 3.3 will indeed always be linearly independent.

OVER

Wronskians

The textbook gives a method for checking whether or not $y_1, \ldots, y_n$ are linearly independent, by using the Wronskian. This is a clever method that relies on matrix algebra and the Existence and Uniqueness Theorem. But we will not cover it, preferring instead to emphasize the *meaning* of linear independence, like in the Questions above.