Study guide for exam 2, math 286, Spring 2009

Best way to study is to go over as many problems as you can in the book. The sections that will be covered are 5.6, 3.8, 9.1, 9.2, 9.3, 9.4, 9.5, 9.6. You may also be asked questions relating to the IODE project! Here are sample solved problems. The actual problems on the exam may be different from these.

Let

$$f(x) = \begin{cases} 0 & \textbf{if } -1 < x < 0 \\ x & \textbf{if } 0 \leq x < 1 \end{cases}$$

extended periodically.

a) Compute the Fourier series for f.

We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$, where (for $n \geq 1$)

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx = \int_{-1}^{0} 0 \cos n\pi x \, dx + \int_{0}^{1} x \cos n\pi x \, dx = \Big[\frac{x}{n\pi} \sin n\pi x\Big]_{x=0}^{1} - \int_0^1 \frac{1}{n\pi} \sin n\pi x \, dx$$
$$= \frac{1}{n^2\pi^2} \big[\cos n\pi x\big]_{x=0}^{1} = \frac{(-1)^n - 1}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-2}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

and

$$a_0 = \int_{-1}^{1} f(x) \, dx = \int_0^1 x \, dx = \frac{1}{2}$$

and

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx = \int_{-1}^{0} 0 \sin n\pi x \, dx + \int_0^1 x \sin n\pi x \, dx$$
$$= \left[\frac{-x}{n\pi} \cos n\pi x\right]_{x=0}^{1} + \int_0^1 \frac{1}{n\pi} \cos n\pi x \, dx = \frac{(-1)^{n+1}}{n\pi}$$

b) Write the series explicitly up to the 3rd harmonic.

$$\frac{1}{4} - \frac{2}{\pi^2} \cos \pi x + \frac{1}{\pi} \sin \pi x - \frac{1}{2\pi} \sin 2\pi x - \frac{2}{9\pi^2} \cos 3\pi x + \frac{1}{3\pi} \sin 3\pi x + \cdots$$

Use separation of variables to find a nontrivial solution to $u_{xx} + u_{yy} = 0$ where $u(x,0) = 0$ and $u(0,y) = 0$. *Hint: Try $u(x,y) = X(x)Y(y)$.*

Try: $u(x,y) = X(x)Y(y)$. Plug into the equation to get

$$0 = u_{xx} + u_{yy} = X''Y + XY''$$

So

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

where the left hand side depends only on x and the right hand side depends only on y. Hence they are both constant. For some constant λ such that

$$\lambda = \frac{X''}{X} = -\frac{Y''}{Y}$$

So we have $X'' - \lambda X = 0$ and $Y'' + \lambda Y = 0$. $u(x,0) = 0$ implies that $Y(0) = 0$ and $u(0,y) = 0$ implies that $X(0) = 0$.

We are only looking for one nontrivial solution so let's assume that $\lambda = 1$. Then there are solutions $X(x) = \sinh x$ and $Y(y) = \sin(y)$, that satisfy $X(0) = Y(0) = 0$. So one nontrivial solution is

$$u(x,y) = (\sinh x)(\sin y)$$

(Note that it is not the unique solution). You might also get say $(\sin x)(\sin y)$ or any linear combination of those two.

Let $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin n\pi t$. Find the steady periodic solution to $x'' + x' + x = f(t)$. Express your solution as a fourier series. You do not need to simplify your expressions for the coefficients.

Try $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t$. Note that there are no conflicts. This is because the complementary solution is damped and hence the terms in the complementary solution contain exponentials.

So,

$$f(t) = x'' + x' + x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (-n^2\pi^2 a_n + n\pi b_n + a_n) \cos n\pi t + (-n^2\pi^2 b_n - n\pi a_n + b_n) \sin n\pi t$$

Hence

$$\begin{aligned} a_0 &= 0 \\ (1 - n^2\pi^2)a_n + n\pi b_n &= 0 \\ -n\pi a_n + (1 - n^2\pi^2)b_n &= \frac{1}{n^3} \end{aligned}$$

So

$$\begin{aligned} a_n &= \frac{-n\pi}{1 - n^2\pi^2} b_n \\ a_n &= \frac{1}{n^3} + \frac{1 - n^2\pi^2}{n\pi} b_n \end{aligned}$$

and so

$$\frac{-n\pi}{1 - n^2\pi^2} b_n = \frac{1}{n^3} + \frac{1 - n^2\pi^2}{n\pi} b_n$$

and so

$$\left(\frac{-n\pi}{1 - n^2\pi^2} - \frac{1 - n^2\pi^2}{n\pi} \right) b_n = \frac{1}{n^3}$$

and so

$$b_n = \frac{1}{n^3} \left(\frac{-n\pi}{1 - n^2\pi^2} - \frac{1 - n^2\pi^2}{n\pi} \right)^{-1}$$

and so

$$a_n = \frac{-n\pi}{(1 - n^2\pi^2)n^3} \left(\frac{-n\pi}{1 - n^2\pi^2} - \frac{1 - n^2\pi^2}{n\pi} \right)^{-1}$$

Find the Fourier series coefficients of the even periodic extension of the function $f(t) = t^2$ for $0 \leq t \leq \pi$.

We will write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} t^2 \, dt = \frac{2\pi^2}{3}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} t^2 \cos nt \, dt = \frac{2}{\pi} \left[t^2 \frac{1}{n} \sin nt \right]_{t=0}^{\pi} - \frac{4}{n\pi} \int_0^{\pi} t \sin nt \, dt \\ &= \frac{4}{n^2\pi} [t \cos nt]_{t=0}^{\pi} + \frac{4}{n^2\pi} \int_0^{\pi} \cos nt \, dt = \frac{4(-1)^n}{n^2} \end{aligned}$$

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a) Let $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos nt$. Is f continuous and differentiable everywhere? Find the derivative (if it exists).

Yes f is continuous and differentiable and the derivative is

$$f'(t) = \sum_{n=1}^{\infty} \frac{-1}{n^2} \sin nt$$

b) Let $f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nt$. Is f differentiable everywhere? Find the derivative (if it exists).

No f is not continuous and differentiable everywhere. If it were the derivative would have been

$$\sum_{n=1}^{\infty} (-1)^n \cos nt$$

but that series does not converge. *It may be (and it is in fact) differentiable at many points, but there must exist at least one point where the function is not differentiable (it in fact has a jump discontinuity).*

Using the d'Alembert solution solve $u_{tt} = 4u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin x$ and $u_t(x,0) = \sin x$.

An odd 2π periodic extension of $\sin x$ is just $\sin x$, so $F(x) = G(x) = \sin x$. Also $a = 2$ because $a^2 = 4$. Now the solution is always of the form

$$u(x,t) = A(x - at) + B(x + at)$$

for the right A and B. To satisfy the initial conditions we write

$$\begin{aligned} A(x) &= \frac{1}{2}F(x) - \frac{1}{2a}\int_0^x G(s)\ ds \\ B(x) &= \frac{1}{2}F(x) + \frac{1}{2a}\int_0^x G(s)\ ds \end{aligned}$$

Note that $\int_0^x G(s)\ ds = 1 - \cos x$. So

$$u(x,t) = \frac{1}{2}\sin(x-2t) - \frac{1}{4}(1-\cos(x-2t)) + \frac{1}{2}\sin(x+2t) + \frac{1}{4}(1-\cos(x+2t)).$$

Now check that this works. First note that (as sin is odd and cos is even)

$$\begin{aligned} u(0,t) &= \frac{1}{2}\sin(-2t) - \frac{1}{4}(1-\cos(-2t)) + \frac{1}{2}\sin(2t) + \frac{1}{4}(1-\cos(2t)) \\ &= \frac{-1}{2}\sin(2t) - \frac{1}{4}(1-\cos(2t)) + \frac{1}{2}\sin(2t) + \frac{1}{4}(1-\cos(2t)) = 0. \end{aligned}$$

Similarly (as $\sin(\pi + \omega) = -\sin(\omega)$)

$$\begin{aligned} u(\pi,t) &= \frac{1}{2}\sin(\pi-2t) - \frac{1}{4}(1-\cos(\pi-2t)) + \frac{1}{2}\sin(\pi+2t) + \frac{1}{4}(1-\cos(\pi+2t)) \\ &= \frac{-1}{2}\sin(-2t) + \frac{1}{4}(1-\cos(-2t)) - \frac{1}{2}\sin(+2t) - \frac{1}{4}(1-\cos(+2t)) = 0. \end{aligned}$$

Moving on,

$$u(x,0) = \frac{1}{2}\sin(x) - \frac{1}{4}(1-\cos(x)) + \frac{1}{2}\sin(x) + \frac{1}{4}(1-\cos(x)) = \sin x$$

and

$$u_t(x,t) = -\cos(x-2t) + \frac{1}{2}\sin(x-2t) + \cos(x+2t) + \frac{1}{2}\sin(x+2t)$$

so

$$u_t(x,0) = -\cos(x) + \frac{1}{2}\sin(x) + \cos(x) + \frac{1}{2}\sin(x) = \sin x.$$

Yay!

Suppose A is a matrix with eigenvalues -1, 1, and corresponding eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$.

a) Find the matrix A

Let $E = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ be the matrix of eigenvectors. We then know that $A = EDE^{-1}$, where $D = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$. We find $E^{-1} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$. Then

$$A = EDE^{-1} = \begin{bmatrix}-1 & 0\\-2 & 1\end{bmatrix}$$

.

You should check that $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$ and You should check that $A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

b) Find the fundamental solution to $\vec{x}' = A\vec{x}$

The fundamental solution is e^{At} and we calculate this with (D and E as in part a))

$$e^{At} = Ee^{Dt}E^{-1} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\0 & e^{t}\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} = \begin{bmatrix}e^{-t} & 0\\e^{-t}-e^{t} & e^{t}\end{bmatrix}$$

c) Solve the system in b) with initial conditions $\vec{x}(0) = \begin{bmatrix}2\\3\end{bmatrix}$

The solution is

$$e^{At}\vec{x}(0) = \begin{bmatrix}e^{-t} & 0\\e^{-t}-e^{t} & e^{t}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2e^{-t}\\2e^{-t}+e^{t}\end{bmatrix}$$

You should check that this really solves the problem.

Using same matrix A as in last problem solve $\vec{x}' = A\vec{x} + \vec{F}$ where $F(t) = \begin{bmatrix}e^{t}\\t\end{bmatrix}$ for $\vec{x}(0) = \begin{bmatrix}2\\3\end{bmatrix}$

Note that we have the eigenvalues and eigenvectors of A computed. So We are looking for a solution $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}g_1 + \begin{bmatrix}0\\1\end{bmatrix}g_2$. We wish to also write f in terms of the eigenvectors. That is we wish to write $\vec{F} = \begin{bmatrix}e^{t}\\t\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}f_1 + \begin{bmatrix}0\\1\end{bmatrix}f_2$. So

$$\begin{bmatrix}f_1\\f_2\end{bmatrix} = E^{-1}\begin{bmatrix}e^{t}\\t\end{bmatrix} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}e^{t}\\t\end{bmatrix} = \begin{bmatrix}e^{t}\\t-e^{t}\end{bmatrix}$$

So $f_1 = e^{t}$ and $f_2 = t - e^{t}$.

Now we also want to write $\vec{x}(0)$ in terms of the eigenvectors. I.e. That is, we wish to write $\vec{x}(0) = \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}a_1 + \begin{bmatrix}0\\1\end{bmatrix}a_2$. So

$$\begin{bmatrix}a_1\\a_2\end{bmatrix} = E^{-1}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

So $a_1 = 2$ and $a_2 = 1$. We plug in our $\vec{x}$ to the equation and get that

$$\begin{aligned}\begin{bmatrix}1\\1\end{bmatrix} g_1' + \begin{bmatrix}0\\1\end{bmatrix} g_2' &= A\begin{bmatrix}1\\1\end{bmatrix} g_1 + A\begin{bmatrix}0\\1\end{bmatrix} g_2 + \begin{bmatrix}1\\1\end{bmatrix} f_1 + \begin{bmatrix}0\\1\end{bmatrix} f_2 \\ &= \begin{bmatrix}1\\1\end{bmatrix}(-g_1) + \begin{bmatrix}0\\1\end{bmatrix} g_2 + \begin{bmatrix}1\\1\end{bmatrix} e^t + \begin{bmatrix}0\\1\end{bmatrix}(t - e^t)\end{aligned}$$

So we get the two equations

$$\begin{aligned} g_1' &= -g_1 + e^t \quad \text{where } g_1(0) = a_1 = 2 \\ g_2' &= g_2 + t - e^t \quad \text{where } g_2(0) = a_2 = 1\end{aligned}$$

We solve with integrating factor

$$g_1 = e^{-t}\int e^t(e^t)\,dt + C_1e^{-t} = \frac{e^t}{2} + C_1e^{-t}$$

And as $g_1(0) = 2$ we have that $2 = \frac{1}{2} + C_1$ and hence $C_1 = \frac{3}{2}$.

Similarly

$$g_2 = e^t\int e^{-t}(t - e^t)\,dt + C_2e^t = e^t\int te^{-t}dt - e^t\int 1dt + C_2e^t = -t - 1 - te^t + C_2e^t$$

And as $g_2(0) = 1$ we have that $1 = -1 + C_1$ and hence $C_2 = 2$.

Hence the solution is

$$\vec{x}(t) = \begin{bmatrix}1\\1\end{bmatrix}\left(\frac{e^t}{2} + \frac{3}{2}e^{-t}\right) + \begin{bmatrix}0\\1\end{bmatrix}\left(-t - 1 - te^t + 2e^t\right) = \begin{bmatrix}\frac{e^t}{2} + \frac{3}{2}e^{-t}\\ \frac{5}{2}e^t + \frac{3}{2}e^{-t} - t - 1 - te^t\end{bmatrix}$$

Find eigenvalues and eigenfunctions of the problem $x'' + \lambda x = 0$, $x'(0) = 0$, $x(\pi) = 0$.

First, let's tackle $\lambda < 0$. Then the general solution is

$$x(t) = A\cosh\sqrt{-\lambda}\,t + B\sinh\sqrt{-\lambda}\,t.$$

From this we get

$$x'(t) = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}\,t + B\sqrt{-\lambda}\cosh\sqrt{-\lambda}\,t.$$

Hence,

$$0 = x'(0) = 0 + B\sqrt{-\lambda}$$

So $B = 0$ since $\sqrt{-\lambda} \neq 0$. Now

$$0 = x(\pi) = A\cosh\sqrt{-\lambda}\,\pi,$$

implies $A = 0$, since $\cosh\sqrt{-\lambda}\,\pi \neq 0$ (cosh is only zero at zero). So there are no negative eigenvalues.

Now let's check $\lambda = 0$. The general solution here is $x(t) = Ax + B$. $0 = x'(0) = A$, so $A = 0$. Now $0 = x(\pi) = B$, so $B = 0$. $\lambda = 0$ is not an eigenvalue.

Finally let $\lambda > 0$. The general solution is

$$x(t) = A\cos\sqrt{\lambda}\,t + B\sin\sqrt{\lambda}\,t.$$

and

$$x'(t) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\,t + B\sqrt{\lambda}\cos\sqrt{\lambda}\,t.$$

Hence $0 = x'(0) = B\sqrt{\lambda}$ so $B = 0$. Next $0 = x(\pi) = A\cos\sqrt{\lambda}\pi = 0$ implies that $\cos\sqrt{\lambda}\pi = 0$ which is true when

$$\sqrt{\lambda}\pi = n\pi + \frac{\pi}{2}$$

for an integer $n = 1, 2, 3, \ldots$ Hence the eigenvalues are

$$\left(n\pi + \frac{\pi}{2}\right)^2$$

and the corresponding eigenfunctions are

$$\cos\left(n\pi + \frac{\pi}{2}\right)t.$$