

Short study guide for final, math 286, Fall 2008

Note that the final is comprehensive, so study all your study guides and past exams and exam solution sheets. You may also be asked questions relating to the IODE project! Here are some sample solved problems from sections 9.7, 10.1, 10.2, 10.3 only.

Let R be the region described by $0 < x < \pi$ and $0 < y < \pi$. Solve the problem

$$\Delta u = 0, \quad u(x, 0) = \sin x, \quad u(x, \pi) = 0, \quad u(0, y) = 0, \quad u(\pi, y) = 0.$$

We note that $(\sin nx)(\sinh n(\pi - y))$ are solutions (see formula (13) on pg. 647 in book). If you want to figure out these solutions again, try $u(x, y) = X(x)Y(y)$, and plug in, you will get an eigenvalue problem $X'' + \lambda X = 0$, with $X(0) = 0$ and $X(\pi) = 0$. You will find out that λ must be positive, and in fact lambda must be $\lambda_n = n^2$. Then you find that your eigenfunction must be $\sin nx$. For Y you get the problem $Y'' - \lambda Y = 0$ with $Y(\pi) = 0$. Hence $X_n = \sin nx$ and $Y_n = \sinh n(\pi - y)$. So your corresponding solutions to $\Delta u = 0$ are $(u_n(x, y) = \sin nx)(\sinh n(\pi - y))$.

Then we write a solution

$$\sum_{n=1}^{\infty} c_n (\sin nx)(\sinh n(\pi - y))$$

where c_n are the sine series coefficients of $f(x) = y(x, 0) = \sin x$ divided by $\sinh \pi$. I.e. $c_1 = \frac{1}{\sinh \pi}$ and $c_n = 0$ for $n > 1$. Thus the solution is

$$u(x, y) = \frac{(\sin x)(\sinh(\pi - y))}{\sinh \pi}.$$

Make sure to check that this works!

Let R be the same region as above, but now solve

$$\Delta u = 0, \quad u(x, 0) = x, \quad u(x, \pi) = x, \quad u(0, y) = 0, \quad u(\pi, y) = \pi.$$

Hint: Try a solution of the form $u(x, y) = X(x) + Y(y)$ (different separation of variables).

OK so try $u = X + Y$. Hence $X'' + Y'' = 0$, that is $X'' = -Y'' = \lambda$. Hence $X = \frac{\lambda}{2}x^2 + Bx + C$ and $Y = -\frac{\lambda}{2}y^2 + Dy + E$. Now start applying the conditions. $u(x, 0) = x$ implies that

$$x = X(x) + Y(0) = \frac{\lambda}{2}x^2 + Bx + C + E$$

implies that $\lambda = 0$, $B = 1$ and $C = -E$. Now (with those substitutions) we also must have

$$x = X(x) + Y(\pi) = x - E + D\pi + E = x + D\pi$$

Hence $D = 0$. Thus we already have that $u(x, y) = X(x) + Y(y) = x$. We just need to check that $u(0, y) = 0$ and $u(\pi, y) = \pi$.

Think about what this solution looks like! Think about the conditions and see what your intuition would tell you about the solution. Does this agree with the solution above.

Find eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) - y'(0) = 0, \quad y(1) = 0$$

This is a regular Sturm-Liouville problem so for each eigenvalue we only need to find a single eigenfunction. It turns out that you only need to look for nonnegative eigenvalues by the theorem from class, but let's do that explicitly so you can see what you have to do without the theorem.

First suppose $\lambda < 0$, Let $\alpha = \sqrt{-\lambda}$ then the general solution is

$$y(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

If we plug in the first boundary condition we get

$$0 = y(0) - y'(0) = A + B - \alpha(A - B)(1 + \alpha)A + (1 - \alpha)B$$

From the second equation we get

$$0 = y(1) = Ae^{\alpha} + Be^{-\alpha}$$

or $B = -Ae^{2\alpha}$. Plug this into the first equation to get

$$0 = A\left((1 + \alpha) - (1 - \alpha)e^{2\alpha}\right)$$

Either A is 0 or $(1 + \alpha) - (1 - \alpha)e^{2\alpha}$ must be zero. This is true at $\alpha = 0$, but if we let $f(\alpha) = (1 + \alpha) - (1 - \alpha)e^{2\alpha}$ and we compute $f'(\alpha) = 1 + (2\alpha - 1)e^{2\alpha}$ we see that $f'(\alpha) > 0$ for $\alpha > 0$ and hence $f(\alpha) > 0$ for $\alpha > 0$. Thus $A = 0$.

Now suppose that $\lambda = 0$. Your general solution is $y(x) = Ax + B$. $y(1) = 0$ implies $A + B = 0$, $y(0) - y'(0) = 0$ implies $B - A = 0$, and hence we must get that $A = B = 0$ hence $\lambda = 0$ is not an eigenvalue.

Thus assume that $\lambda > 0$. Your general solution is then

$$y(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x$$

Applying the conditions we get

$$0 = y(0) - y'(0) = A - \sqrt{\lambda}B \quad 0 = y(1) = A \cos \sqrt{\lambda} + B \sin \sqrt{\lambda}$$

That is $A = \sqrt{\lambda}B$ and hence unless $A = B = 0$ we must have that

$$\cos \sqrt{\lambda} + \sqrt{\lambda} \sin \sqrt{\lambda} = 0$$

This means that

$$\frac{1}{\sqrt{\lambda}} = -\tan \sqrt{\lambda}$$

Hence if β_n are the positive solutions to

$$\frac{1}{\beta} = -\tan \beta$$

then $\lambda_n = \beta_n^2$. The corresponding eigenfunctions are

$$y_n(x) = \cos \beta_n x + \beta_n \sin \beta_n x$$

Suppose that you had a Sturm-Liouville problem on $[0, 1]$ and came up with $y_n(x) = \sin \gamma_n$, where $\gamma > 0$ is some constant. Decompose $f(x) = x$, $0 < x < 1$ in terms of these eigenfunctions.

Well, according to the formula $c_n = \langle f, y_n \rangle = \langle y_n, y_n \rangle$, we compute

$$c_n = \frac{\int_0^1 x y_n(x) dx}{\int_0^1 (y_n(x))^2 dx}$$

First compute

$$\int_0^1 (y_n(x))^2 dx = \int_0^1 (\sin \gamma_n x)^2 dx = \frac{2\gamma_n - \sin(2\gamma_n)}{4\gamma_n}$$

And then

$$\begin{aligned}\int_0^1 f(x)y_n(x)^2 dx &= \int_0^1 x (\sin \gamma n x) dx \\ &= \frac{\sin(\gamma n) - \gamma n \cos(\gamma n)}{\gamma^2 n^2}\end{aligned}$$

So

$$c_n = \frac{\int_0^1 xy_n(x) dx}{\int_0^1 (y_n(x))^2 dx} = \frac{-4(\sin(\gamma n) - \gamma n \cos(\gamma n))}{\gamma n (\sin(2\gamma n) - 2\gamma n)}$$

Then the decomposition is, for $0 < x < 1$ we have

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sum_{n=1}^{\infty} \frac{-4(\sin(\gamma n) - \gamma n \cos(\gamma n))}{\gamma n (\sin(2\gamma n) - 2\gamma n)} \sin \gamma n x$$

Suppose you have a beam of length 5 with free ends. Let y be the transverse deviation of the beam at position x on the beam ($0 < x < 5$). You know that the constants are such that this satisfies the equation $y_{tt} + 4y_{xxxx} = 0$. Suppose you know that the initial shape of the beam is the graph of $x(5-x)$, and the initial velocity is uniformly equal to 2 (same for each x) in the positive y direction. Set up the equation together with the boundary and initial conditions. Just set up, don't solve.

$$\begin{aligned}y_{tt} + 4y_{xxxx} &= 0 \\ y_{xx}(0, t) &= 0 \\ y_{xxx}(0, t) &= 0 \\ y_{xx}(5, t) &= 0 \\ y_{xxx}(5, t) &= 0 \\ y(x, 0) &= x(5-x) \\ y_t(x, 0) &= 2\end{aligned}$$

Suppose you have a beam of length 5 with one end free and one end fixed (the fixed end is at $x = 5$). Let u be the longitudinal deviation of the beam at position x on the beam ($0 < x < 5$). You know that the constants are such that this satisfies the equation $u_{tt} = 4u_{xx}$. Suppose you know that the initial displacement of the beam is $\frac{x-5}{50}$, and the initial velocity is $\frac{-(x-5)}{100}$ in the positive u direction. Set up the equation together with the boundary and initial conditions. Just set up, don't solve.

$$\begin{aligned}u_{tt} &= 4u_{xx} \\ u_x(0, t) &= 0 \\ u(5, t) &= 0 \\ u(x, 0) &= \frac{x-5}{50} \\ u_t(x, 0) &= \frac{-(x-5)}{100}\end{aligned}$$