Several Complex Variables are Better than One

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Abstract

The theory of complex analysis in one variable is in some sense the calculus student's dream come true. No annoying pathological cases, and things generally just work right. However the zero sets of analytic functions in one variable are downright boring (isolated points). In this talk I will talk about the Hartogs Phenomenon which has a coolness factor at least double that of the Maximum Principle. In particular it will tell us something about how the zero sets of analytic functions behave when we have more than one complex variable. You will also find out what the inhomogeneous d-bar equation is, and how to exhibit solutions in certain cases, which should really come in handy next time you are in a bar and need to show off something more impressive than flipping ten beer coasters at once.

This talk should be accessible to all who have not slept through basic calculus (or at least not slept through most of it).

1 Holomorphic Functions in One Variable

In complex analysis, we are studying holomorphic (some people call them analytic) functions. So let's start with one variable. There are many equivalent ways to define what a holomorphic function is, but the following is what most people use. Firstly let's define the following formal differential operators by identifying $\mathbb{C}$ with $\mathbb{R}^2$ as $z = x + iy$,

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Definition 1.1. Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ be a continuously differentiable function. f is said to be *holomorphic* if it satisfies the *Cauchy-Riemann* equations which can be written as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Examples of holomorphic functions are for example polynomials in z or basic function such as sin, cos or exp. We can also multiply and compose these functions to get yet other holomorphic functions.

Normally if you take any function on $\mathbb{C}$ you might think of it as a function on $\mathbb{R}^2$ as identified above and hence as a function of x and y. However we can also compute that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, so we can just think of it as a function of two complex variables z and $\bar{z}$ (pretending almost as if $\bar{z}$ did not depend on z). Then a holomorphic function is a function which does not depend on $\bar{z}$.

Another way to look at holomorphic functions is by their power series expansions. All holomorphic functions have convergent power series in z (converging to the function) at every point and conversely every convergent power series in z defines a holomorphic function. Being holomorphic is also equivalent to the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z_0) - f(z)}{z_0 - z}$$

existing for all $z_0 \in U$. Which of course is the calculus definition of the derivative. Note however that the limit is over a complex variable and thus there are many more ways to approach z_0, which is why it is much more restrictive than having a real derivative. Holomorphic functions in one variable have many wonderful properties which we will not have time to go into here.

When one has a class of functions to play with, a natural problem to look at, is to solve equations. That means looking at the set $\mathcal{Z}_f := \{z \in U : f(z) = 0\}$, which we call the zero set of f. This problem turns out to be somewhat boring in one dimension as the set consists of isolotated points. As $\mathcal{Z}_f$ is a closed set, it means that it has no limit points inside U, though it can of course have limit points on the boundary. It also turns out that for any set of isolated points with no limit points inside U we can find a function that vanishes precisely at those points. Dreadfully boring.

2 Several Variables

So how can we make this a more interesting problem? Let's add more complex variables. Of course we first need to define what do we mean by holomorphic functions. The definition could not be simpler. As we will be in $\mathbb{C}^n$, we will use the notation $z = (z_1, \ldots, z_n)$.

Definition 2.1. Let $U \subset \mathbb{C}^n$ be an open set and let $f\colon U \to \mathbb{C}$ be a function. It is said to be *holomorphic* if the function

$$\zeta \mapsto f(z_1, \ldots, z_{k-1}, \zeta, z_{k+1}, \ldots, z_n)$$

is holomorphic as a function of one variable for all k.

Examples are similar again except now we allow n variables instead of one, but we can still build up functions as before. Further we also have the power series representation of functions. For example in two variables z and w a power series at the origin may look like

$$\sum_{j,k=0}^{\infty} a_{jk} z^j w^k$$

where a_{jk} are complex numbers.

One important property of holomorphic functions arising from the power series representation is that if a holomorphic function f is zero on an open subset of U (where f is defined) then the function is identically zero everywhere in U (provided U is connected). This is of course true in one dimension as well. It gives us some restriction on what the set $\mathcal{Z}_f$ can look like, but also gives us another property. If f is defined on U and has an extension to a larger domain $\tilde{U}$ where $U \subset \tilde{U}$, that is, $\tilde{f}$ is holomorphic on $\tilde{U}$ and is equal to f on U, then this extension is unique provided for example if both U and $\tilde{U}$ are connected. To see this, note that if we have another extension on $\tilde{U}$, say $\hat{f}$, then $\hat{f} - \tilde{f}$ is holomorphic on $\tilde{U}$ and identicaly zero on U.

3 Hartogs' Phenomenon

Let's state the Hartogs' phenomenon theorem which will tell us something even more fascinating about extending functions and further about the topology of zero sets of holomorphic functions.

Theorem 3.1. *Let $n \geq 2$, $U \subset \mathbb{C}^n$ be an open connected set (a domain), and let $K \subset U$ be a compact set (closed and bounded) such that $U \setminus K$ is* *connected. Further suppose there exists a holomorphic function f defined on $U \setminus K$. Then f uniquely extends to a holomorphic function on all of U*

This result is not true in one complex dimension. To see this suppose that we have U be a neighbourhood of the origin and suppose $K = 0$. Then take the function $1/z$ which is holomorphic on $U \setminus K$, but does not extend to U since it blows up near the origin (has a pole).

So why does this tell us something about how zero sets of holomorphic functions look? Well suppose that U and K are as in the theorem and suppose there would be a function f which has K as its zero set. Then $\frac{1}{f}$ would be a holomorphic function on $U \setminus K$ but would blow up (go to infinity) as you approach K so would certainly not be possible to extend through K. So K cannot be the zero set of f nor can the zero set of f be contained in K. With a little bit of fiddling around you can prove that $\mathcal{Z}_f$ cannot be any compact set as long as $n \geq 2$. So this means that for example that $\mathcal{Z}_f$ can never be bounded. It always somehow has to "escape" towards infinity. So in some sense zero sets of holomorphic functions in more then one variable are in some sense "large."

So for example if we have a holomorphic function in more then one variable defined in an open set minus a point, we can just take extend the function by continuity. We just stick in a limit and we don't need to check that it exists, it just will.

Another way that this is suprising, is if you have only looked at the single variable case. There, for any connected open set a function can be found which does not extend into any larger connected open set. So any open set is a natural domain of definition for some holomorphic function. In several complex variables on the other hand, only some open sets are natural domains of definitions of holomorphic functions.

OK, let's look at a sketch of the proof. What we will do is solve a certain differential equation and use the solution to construct the needed extension. First let's see the equation (we'll get to details in a second)

$$\bar{\partial}\psi = g$$

This is the *inhomogeneous* $\bar{\partial}$ (pronounced d-bar) equation. OK, it's really not easy to see what we mean above. In detail this equation means that ψ

is some smooth (not holomorphic) function, and g is a differential form, but we can just think about it being n different smooth functions, g_1 through g_n. Then the above is really n equations

$$\frac{\partial \psi}{\partial \bar{z}_k} = g_k,$$

where are date g_k satisfy the compatibility condition

$$\frac{\partial g_k}{\partial \bar{z}_l} = \frac{\partial g_l}{\partial \bar{z}_k},$$

which is needed basically because partial derivatives are supposed to commute. For our purposes let's now assume that all the g_k have compact support. Then these equations always have a solution ψ (many solutions in fact), and if $n \geq 2$, then we can find ψ which will have compact support. A solution can be obtained by

$$\psi(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}. \tag{1}$$

Where $d\zeta \wedge d\bar{\zeta}$ is just a fancy way to write the area measure on $\mathbb{C}$ up to a funny constant. This will have compact support since g_1 has compact support and as z goes off towards infinity $\zeta, z_2, \ldots, z_n$ also goes off to infinity no matter what ζ is. It should be clear why it doesn't if $n = 1$. One might be wondering why we picked z_1 and g_1 in the above equation. We can actually pick any of them (say g_k and z_k) not just the first one (and get the same function). Further one might wonder why the heck we only use one part of the data, and still get an actual solution. The answer here is that the compatibility condition relates all the data, so we only need to look at one. To check that this really is a solution is left to the reader (You need to apply the generalized Cauchy formula).

Now onto the actual proof of the theorem in a rather handwavy fashion. First let's find a smooth function φ which is 1 in a neighbourhood of K and is compactly supported in U. Now let $f_0 := (1-\varphi)f$ which is now identically zero on K and holomorphic near the boundary of U (since there φ is 0). So now we let $g = \bar{\partial} f_0$, that is $g_k = \frac{\partial f_0}{\partial \bar{z}_k}$. Let's see why g_k is compactly supported. The only place to check is on $U \setminus K$ as elsewhere we have 0 automatically,

$$\frac{\partial f_0}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}((1-\varphi)f) = -f \frac{\partial \varphi}{\partial \bar{z}_k}.$$

And now we apply the above solution (1) to find a compactly supported function ψ such that $\bar{\partial}\psi = g$. Now we just set $\tilde{f} := f_0 - \psi$. Let's check that this is the desired extention. Firstly let's check it's holomorphic,

$$\frac{\partial \tilde{f}}{\partial \bar{z}_k} = \frac{\partial f_0}{\partial \bar{z}_k} - \frac{\partial \psi}{\partial \bar{z}_k} = g_k - g_k = 0$$

It almost feels like we're cheating. OK, but does it extend f. A bit of thought and the fact that $U \setminus K$ is connected will reveal that ψ must be compactly supported in U. This means that $\tilde{f}$ agrees with f near the boundary (in particular on an open set) and thus everywhere in U since U is connected.

Phew ... any questions?

And they lived happily ever after. The End.

References

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