

Some problems to solve with solutions ...

1. Let C be the curve $x = \cos t$, $y = \sin t$, $z = t$, for $0 \leq t \leq 2\pi$ oriented in the direction of increasing t . Evaluate:

$$\int_C z \, dx + x \, dy + y \, dz$$

Solution: First draw a picture of the curve to see what we are integrating over.

We differentiate to get $dx = -\sin t \, dt$, $dy = \cos t \, dt$, $dz = dt$.

$$\begin{aligned} \int_C z \, dx + x \, dy + y \, dz &= \int_0^{2\pi} \left((t)(-\sin t) + (\cos t)(\cos t) + (\sin t) \right) dt \\ &= \int_0^{2\pi} -t \sin t \, dt + \int_0^{2\pi} \cos^2 t \, dt + \int_0^{2\pi} \sin t \, dt \\ &= \left[-\sin t + t \cos t \right]_{t=0}^{2\pi} + \left[\frac{1}{2}t + \frac{1}{4} \sin 2t \right]_{t=0}^{2\pi} + 0 \\ &= 2\pi + \pi + 0 = 3\pi \end{aligned}$$

2. Let S be the unit sphere, $x^2 + y^2 + z^2 = 1$. Let \mathbf{n} be the outward unit normal. Evaluate the following surface integral using Gauss's theorem (the divergence theorem).

$$\iint_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy$$

Solution: First draw a picture. Let S be the sphere and let R be the interior of the sphere. Note that the vector field $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is continuous and has continuous partial derivatives everywhere and so we can apply Gauss's theorem.

$$\begin{aligned} \iint_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy &= \iiint_R (1 + 1 + 1) \, dx \, dy \, dz \\ &= 3 \iiint_R dx \, dy \, dz \\ &= 3(\text{"volume of } R\text{"}) \\ &= 3\left(\frac{4}{3}\pi\right) = 4\pi \end{aligned}$$

3. Let $F(x, y, z) = 2x^2 - y^2 - z^2$. Let R be the unit cube, $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. Let the surface S be the boundary of R . Let \mathbf{n} be the outward unit normal. Evaluate the following surface integral.

$$\iint_S \nabla F \cdot \mathbf{n} \, d\sigma$$

Solution: First draw a picture. Note that the vector field ∇F is continuous and has continuous partial derivatives everywhere and so we can apply Gauss's theorem. Also note that $\operatorname{div} \nabla F = \nabla^2 F$ (the Laplacian of F).

$$\begin{aligned} \iint_S \nabla F \cdot \mathbf{n} \, d\sigma &= \iiint_R \operatorname{div} \nabla F \, dx \, dy \, dz \\ &= \iiint_R \nabla^2 F \, dx \, dy \, dz \\ &= \iiint_R (4 - 2 - 2) \, dx \, dy \, dz \\ &= \iiint_R 0 \, dx \, dy \, dz = 0 \end{aligned}$$

Notice that for any harmonic function F (recall $\nabla^2 F = 0$) we have

$$\iint_S \nabla F \cdot \mathbf{n} \, d\sigma = 0$$

4. Let S be the upper hemisphere of the unit sphere. Let \mathbf{n} be the upper unit normal. Let $\mathbf{v} = zx\mathbf{i} + zy\mathbf{j} + y\mathbf{k}$. Evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

Solution: Let R_{xy} be the region $x^2 + y^2 \leq 1$ in the xy -plane. Notice that S is the graph $z = \sqrt{1 - x^2 - y^2}$. Draw a picture.

First compute

$$\frac{\partial z}{\partial x} = \frac{-2x}{2\sqrt{1-x^2-y^2}} = \frac{-x}{\sqrt{1-x^2-y^2}} = \frac{-x}{z}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2\sqrt{1-x^2-y^2}} = \frac{-y}{\sqrt{1-x^2-y^2}} = \frac{-y}{z}$$

$$\begin{aligned} \iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma &= \iint_S zx \, dy \, dz + zy \, dz \, dx + y \, dx \, dy \\ &= \iint_{R_{xy}} \left[-zx \left(\frac{-x}{z} \right) - zy \left(\frac{-y}{z} \right) + y \right] dx \, dy \\ &= \iint_{R_{xy}} [x^2 + y^2 + y] dx \, dy \\ &= \int_0^{2\pi} \int_0^1 [(r \cos \theta)^2 + (r \sin \theta)^2 + r \sin \theta] r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (r^3 + r^2 \sin \theta) \, dr \, d\theta \\ &= 2\pi \int_0^1 r^3 \, dr + \int_0^1 r^2 \left[\int_0^{2\pi} \sin \theta \, d\theta \right] dr \\ &= 2\pi \frac{1}{4} + 0 = \frac{\pi}{2} \end{aligned}$$

5. Let S be the whole unit sphere. Let \mathbf{n} be the outward unit normal. Let $\mathbf{v} = zx\mathbf{i} + zy\mathbf{j} + y\mathbf{k}$ (as in the last question). Evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

Use this and result of problem 4 to find $\iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma$ where L is the lower hemisphere of S and \mathbf{n} the lower unit normal (hence the same normal as in the surface integral over all of S).

Solution: Again we apply Gauss's theorem. Let R be the interior of S .

$$\begin{aligned} \iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma &= \iiint_R \operatorname{div} \mathbf{v} \, dx \, dy \, dz \\ &= \iiint_R z + z \, dx \, dy \, dz \\ &= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 2z \, dz \, dx \, dy \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 0 \, dz \, dx \, dy \quad (\text{an integral of an odd function over a symmetric interval equals 0}) \\ &= 0 \end{aligned}$$

We let U denote the upper hemisphere. Notice that

$$0 = \iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma = \iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma + \iint_U \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

Hence

$$\iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma = - \iint_U \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

We have computed in problem 4 that $\iint_U \mathbf{v} \cdot \mathbf{n} \, d\sigma = \frac{\pi}{2}$. Hence

$$\iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma = -\frac{\pi}{2}.$$