HW13, DUE DEC 9

Exercise 1) Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function. Find $\lim_{n\to\infty} \int_0^1 \frac{f(x)}{n}\, dx$.

Exercise 2) Let $f_n(x) = \frac{x^n}{n}$. Show that f_n converges uniformly to a differentiable function f on $[0,1]$ (find f). However, show that $f'(1) \neq \lim_{n\to\infty} f_n'(1)$.

Exercise 3) Find an example of a sequence of continuous functions on $(0,1)$ that converges pointwise to a continuous function on $(0,1)$, but the convergence is not uniform.

Exercise 4) Find an example of a sequence of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly to some f and g on some set A, but such that $f_n g_n$ (the multiple) does not converge uniformly to fg on A. Hint: Let $A = \mathbb{R}$, let $f(x) = g(x) = x$. You can even pick $f_n = g_n$.

Exercise 5) Let $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be sequences of functions on $[a,b]$. Suppose that f_n and h_n converge uniformly to some function $f\colon [a,b] \to \mathbb{R}$ and suppose that $f_n(x) \leq g_n(x) \leq h_n(x)$ for all $x \in [a,b]$. Show that g_n converges uniformly to f.

Exercise 6) Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. Suppose that $\int_a^x f = \int_x^b f$ for all $x \in [a,b]$. Show that $f(x) = 0$ for all $x \in [a,b]$.

Exercise 7) Suppose that F, and G are differentiable functions defined on $[a,b]$ such that $F'(x) = G'(x)$ for all $x \in [a,b]$. Show that F and G differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$.