HW9, DUE OCT 28

Exercise 1) Give an example of functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that the function h defined by $h(x) := f(x) + g(x)$ is continuous, but f and g are nowhere continuous (not continuous at any point).

Exercise 2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$ we have $f(x) > 0$.

Exercise 3) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all x.

Exercise 4) Find an example of a *bounded* discontinuous function $f\colon [0,1] \to \mathbb{R}$ which has neither an absolute minimum nor an absolute maximum.

Exercise 5) Suppose $f\colon (0,1) \to \mathbb{R}$ is a continuous function such that $\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0,1)$ (but perhaps not both).

Exercise 6) Suppose that $g(x)$ is a polynomial of even degree d such that

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Suppose that $g(0) < 0$. Show that g has at least two distinct real roots.

Exercise 7) Let

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

Show that f has the intermediate value property. That is, for any $a < b$, and any y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$ there exists a $c \in (a,b)$ such that $f(c) = y$.