1 Introduction

My research lies in the field of commutative algebra, the study of commutative rings and modules. My research focuses on a set of interconnected open problems among the Homological Conjectures. These problems have their origins in the works of Auslander [Aus61], Bass [Bas63], Serre [Ser00], Peskine and Szpiro[PS73], and Hochster [Hoc73]. The Homological Conjectures have important ramifications in algebra and algebraic geometry involving invariant theory, cohomology of vector bundles, the solutions of polynomial equations, characteristic-p techniques and intersection theory. In the following I will outline those problems that are connected with my research.

2 Previous Work

Most of the Homological Conjectures are known in the case where the ring in question contains a field. Many of these conjectures remain open in the mixed characteristic case where the ring does not contain a field, though some partial results are known. Below the most relevant conjectures to my work are outlined. This is not a comprehensive list and many important results are omitted for brevity.

In [Hoc73], Hochster proved the following statement in the case where the ring R contains a field:

Conjecture 1 (The Direct Summand Conjecture). *Let R be a regular local ring and A be a module-finite extension. Then R is a direct summand of A.*

Much work has been done on the Direct Summand Conjecture (henceforth DSC), including Heitmann's proof [Hei02] of the DSC in dimension 3. Many people, including Hochster, Dutta, Evans, Griffith, Koh, Goto and Roberts, have been instrumental in advancing progress on the Homological Conjecture. (See e.g. [Hoc83], [HH95], [Dut01], [Rob76], [EG81], [Got83], [Koh86]) In particular, Hochster reformulated the DSC into equivalent versions, one in terms of certain elements in local cohomology modules called the Canonical Element Conjecture [Hoc83], and one characterizing the DSC in terms of the nonvanishing of certain polynomial equations, called the Monomial Conjecture [Hoc73]. We discuss a stronger version of the Monomial Conjecture in the next section.

In more recent work, Hochster and Huneke [HH95] formulated a stronger conjecture, called the Vanishing Maps of Tor Conjecture (VMTC).

Conjecture 2 (Vanishing Maps of Tor Conjecture(VMTC)). *Let $R \to A \to S$ be homomorphisms of Noetherian rings such that R and S are regular and such that A is module-finite over R. Let M be any R-module. Then the maps*

$$\mathrm{Tor}_i^R(M, A) \to \mathrm{Tor}_i^R(M, S)$$

are zero for all $i \geq 1$.

They showed that this conjecture is also known when the rings in question contain a field [HH95] and later Hochster proved the dimension 3 case [Hoc02], but the question is open in general. This new conjecture not only implies the DSC, but also implies that direct summands of regular rings are Cohen-Macaulay.

Later Ranganathan [Ran00] showed that the VMTC was actually equivalent to the following statement:

Conjecture 3 (Strong Direct Summand Conjecture)**.** *Let R be a regular local ring and let A be a module finite extension. Let Q be a height one prime ideal of A containing xR where x is a minimal generator of the maximal ideal of R. Then xR is a direct summand of Q.*

While proving the equivalence of the VMTC and SDSC, Ranganathan made the following observations. First, the SDSC reduces to the case where R and A are complete local domains. Second, Ranganathan showed that it was sufficient to study the case where $A = R + Q$ and hence $A/Q = R/xR$.

As the name suggests, the Strong Direct Summand Conjecture (SDSC) implies the DSC. Few of the partial results for the DSC have been extended to the SDSC. Moreover, the SDSC provides a new avenue for attacking the VMTC. The SDSC and VMTC are the main focus of my research.

3 Results

My research follows along two directions. First, I would like to state several important special cases of the SDSC that I established. (A preprint [McC08] containing these results is available on my web page at `http://www.math.uiuc.edu/~jmccullo`.)

3.1 Some new cases of the SDSC

Keeping the notation from the statement of the SDSC, we first have the following.

Theorem 3.1. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and let A be a module finite extension of R. Let x be a minimal generator of $\mathfrak{m}$ and let Q be a height one prime of A lying over xR. Suppose that $\operatorname{Ext}^1_R(A, R) = 0$ (equivalently, $H^{d-1}_{\mathfrak{m}}(A) = 0$). Further suppose that $R/xR \to A/Q$ splits. Then the SDSC holds for Q and A; that is, the map $xR \to Q$ splits.*

Above $H^{d-1}_{\mathfrak{m}}(A)$ denotes the $d-1$st local cohomology module of A with respect to the maximal ideal $\mathfrak{m}$ of R.

Let us recall that we may assume that R and A are complete local domains. We note that the condition that $\operatorname{Ext}^1_R(A, R) = 0$ holds in three important cases.

Corollary 3.2. *Let A be a complete local domain of dimension d. Suppose the DSC holds for rings of dimension $d-1$. Then the SDSC holds in the following cases:*

1. *A is Cohen-Macaulay.*
2. *A is an almost complete intersection domain.*
3. *A is normal and ω_A is S_3, where ω_A denotes the canonical module of A.*

We may also assume that $A = R + Q$. In this case, the splitting of $R/xR \to A/Q$ is obvious, since, in this case, $A/Q = R/xR$. Thus we can state the following result without the inductive hypothesis of the previous corollary.

Theorem 3.3. *Let $R \to A$ be a module finite extension of complete local domains. Suppose R is regular. Let x be a minimal generator of the maximal ideal of R and let Q be a height one prime lying over xR in A. Set $B = R + Q$. Then the $xR \to Q$ splits in the following cases:*

1. *B is Cohen-Macaulay.*
2. *B is an almost complete intersection domain.*
3. *B is normal and ω_B is S_3, where ω_B denotes the canonical module of B.*

3.2 The Strong Monomial Conjecture

In the other part of my research, I have considered the following Strong Monomial Conjecture, as stated by Ranganathan in [Ran00]:

Conjecture 4 (Strong Monomial Conjecture(SMC))**.** *Let A be a local domain with system of parameters $x_1, \ldots, x_d$. Let Q be a height one prime of A containing x_1. Then for all $t > 0$,*

$$x_1(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})Q.$$

The SMC implies the SDSC. The following is an extension of work by Dutta [Dut98] and Strooker and Stückrad [SS93] on the Monomial Conjecture.

First, we need the following setup. Recall that we may assume that A is a complete local domain. In this case, we can find a local complete intersection S and a surjection $S \twoheadrightarrow A$ such that $\dim S = \dim A$. Without loss of generality we can assume that every system of parameters of A may be lifted to S. In this situation, we have the following:

Theorem 3.4. *Let $(A, \mathfrak{m})$ be a complete local domain and let $\mathbf{x} = x_1, \ldots, x_d$ a system of parameters for A. Let Q a height one prime of A. Let $z \in Q - \mathbf{x}Q$. Then for all $t > 0$,*

$$z(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})Q$$

if and only if

$$J_z \not\subset \mathbf{x}S,$$

where J_z is the image of $\operatorname{Hom}_S(Q, S)$ in $\operatorname{Hom}_s(S, S) \simeq S$ induced by the map $S \to Q$ where $1 \mapsto z$.
In particular, the strong monomial conjecture holds if and only if

$$J_{x_1} \not\subset \mathbf{x}S.$$

In other words, the SMC holds if and only if the ideal J_{x_1} in S defined above is not contained in any parameter ideal.

Continuing in this direction, I extended other results of Dutta [Dut98] on the MC to the SMC. First, I proved the following reduction of the SMC:

Corollary 3.5. *The Strong Direct Summand Conjecture holds for all local rings if and only if it holds for all local almost complete intersection rings.*

Thus to prove the SMC, we need only prove it for this smaller class of rings.

Again, using the theorem above, I proved another equivalent form of the SMC in terms of Koszul homology modules. Keeping the notation from Theorem 3.4, we have the following:

Theorem 3.6. *The the Strong Monomial Conjecture holds for Q, z and $\mathbf{x}$ if and only if $\ell(J_z/\mathbf{x}J_z) > \ell(H_1(\mathbf{x}; S/J_z))$.*

Here $\ell(-)$ denotes the length of a module and $H_1(\mathbf{x}; S/J_z)$ denotes the first Koszul homology module on S/J_z with respect to the system of parameters $x_1, \ldots, x_d$. For an arbitrary module M, $H_1(\mathbf{x}; M)$ may not be finite length, but using another result of Dutta and Samuel's theory of superficial elements, I proved the following:

Theorem 3.7. *Suppose the residue field of A is infinite. Let $x_1, \ldots, x_d$ be a s.o.p. Let Q be a height one prime in A. Finally let J_{x_1} be as above. Then there exist $y_1, \ldots, y_{d-1} \in A$ such that $(x_1, \ldots, x_d) = (y_1, \ldots, y_{d-1}, x_d)$ and such that SMC holds for A, Q, $y_1, \ldots, y_{d-1}, x_d^t$ and x_1 for $s \gg 0$; that is,*

$$x_1(y_1 \cdots y_{d-1}, x_d^s)^t \notin (y_1^{t+1}, \ldots, y_{d-1}^{t+1}, (x_d^s)^{t+1})Q$$

Thus by altering the system of parameters slightly, we find cases where the SMC and hence SDSC holds.

4 Current and Future Research

The Strong Direct Summand Conjecture and related Vanishing Maps of Tor Conjecture will continue to be a primary focus of my research. Here are a few directions I intend to take my research in the future:

1. Recall that both of these conjectures imply the Direct Summand Conjecture, so any new special cases would immediately translate to new known cases of the Direct Summand Conjecture. For instance, in the case where the regular parameter $x = p$, the characteristic of the residue field of R, the direct summand conjecture is known for R/pR. Thus it may be possible to use characteristic p techniques for this case of the Strong Direct Summand Conjecture.

2. It is often the case that one can lift a splitting in one dimension lower of $R/xR \to A/Q$ to $R \to A$, as is the case when $H^{d-1}_{\mathfrak{m}}(A) = 0$ (i.e. $\mathrm{Ext}^1_R(A, R) = 0$). It would be useful to find other cases where one can lift splittings from one dimension lower.

3. In the proof of the equivalence of the Strong Direct Summand and Vanishing Maps of Tor conjectures, one reduces both to the case where the ring A is of the form $R + Q$ where Q is the prime ideal in A. Thus if one can prove either conjecture in this smaller case, the full conjecture(s) would follow. However, in passing from A to $R + Q$ one potentially loses nice properties of the ring A. I intend to study exactly what properties do pass from A to $R + Q$.

4. Another line of inquiry I plan to continue involves questions about characteristic p rings. In particular, Jinjia Li and Ian Aberbach [AL08] recently proved that a local ring $(A, \mathfrak{m}, k)$ of characteristic p and dimension d is regular if and only if $t_i := \lim_{n\to\infty} \frac{\ell(\mathrm{Tor}_i^R(k, {}^{f^n}A))}{p^{nd}} = 0$ for any (equivalently all) $i \geq 1$. Here ${}^{f^n}A$ represents the A-module A with left module structure given by the nth iterated Frobenius action: $a\dot{x} = a^{p^n}x$. Their proof uses the tight closure and phantom homology theory developed by Hochster and Huneke [HH90], [HH93]. It would be interesting to obtain a proof using direct methods.

   In related work, Avramov, Iyengar and Miller [AIM06] proved that the Betti numbers of ${}^{f^n}A$ grow at the same rate as those of the residue field k (up to a constant). In other words, $\ell(\mathrm{Tor}_i^R(k, {}^{f^n}A)) = O(\ell(\mathrm{Tor}_i^R(k, k)))$. One can ask a similar question about the numbers t_i above; that is, up to a constant, do the Betti numbers of k. Preliminary calculations done in Macaulay 2 suggest this is true. This question is closely related with the previous one as well.

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