On the Strong Direct Summand Conjecture

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Notation

All rings are Noetherian commutative rings with identity.

Vanishing of Maps of Tor Conjecture

Conjecture (Hochster-Huneke '95)

Let $R \hookrightarrow A \to S$ be maps of Noetherian rings such that R and S are regular and A is module-finite over R. Let M be any R-module. Then the maps

$$\mathrm{Tor}_i^R(M, A) \to \mathrm{Tor}_i^R(M, S)$$

are zero for $i \geqslant 1$.

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- Implies the Direct Summand Conjecture.

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- Can assume R, A are complete local domains.

Strong Direct Summand Conjecture

Conjecture (SDSC - Ranganathan '00)

Let $(R, \mathfrak{m})$ be a regular local ring and let A be a module finite extension. Let Q be a height one prime ideal of A containing xR, where $x \in \mathfrak{m} - \mathfrak{m}^2$. Then xR is a direct summand of Q.

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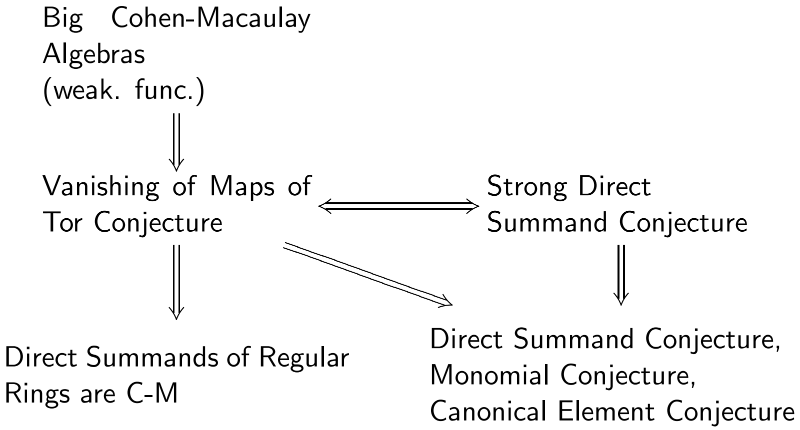
- Implies the Direct Summand Conjecture. (Factor as $xR \longrightarrow xA \longrightarrow Q$.)
- Equivalent to the Vanishing of Maps of Tor Conjecture (Ranganathan).
- True in equicharacteristic or if $\dim(R) \leqslant 3$.
- Can assume R, A are complete local domains and that $A = R + Q$. (Splitting of $xR \hookrightarrow Q$ does not change.)

Diagram of Implications



Proposition (-)

Let R, A, x, Q be as in SDSC and suppose A is a domain. Then the following are equivalent:

1. *There exist retractions $\rho : A \to R$ and $\eta : A/Q \to R/xR$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \longrightarrow & A/Q \\ \downarrow \rho & & \downarrow \eta \\ R & \longrightarrow & R/xR. \end{array}$$

2. *The map $xR \to Q$ splits.*

R regular local
A module-finite over R
$x \in \mathfrak{m} - \mathfrak{m}^2$
$Q \subseteq A$ with $\mathrm{ht}(Q) = 1$.

Theorem (-)

Let R, A, x, Q be as in SDSC. Suppose $H_{\mathfrak{m}}^{d-1}(A) = 0$ and $R/xR \longrightarrow A/Q$ splits. Then $xR \longrightarrow Q$ splits.

- Uses that the Direct Summand Conjecture holds when $H_{\mathfrak{m}}^{d-1}(A) = 0$. (Dutta)

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Theorem (-)

Let A be a complete local domain of dimension d. Suppose the DSC holds for rings of dimension $d-1$. Then the SDSC holds in the following cases:

1. *A is Cohen-Macaulay.*
2. *A is an almost complete intersection domain.*
3. *A is normal and ω_A is S_3, where ω_A denotes the canonical module of A.*

Proof.

$H_{\mathfrak{m}}^{d-1}(A) = 0$ in all 3 cases. ■

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$Q \subseteq A$ with $\mathrm{ht}(Q) = 1$.

Theorem (-)

Let A be a complete local domain of dimension d. Then $xR \to Q$ splits in the following cases:

1. $R + Q$ *is Cohen-Macaulay.*
2. $R + Q$ *is an almost complete intersection domain.*
3. $R + Q$ *is normal and* ω_{R+Q} *is* (S_3).

Note: When $A = R + Q$, $A/Q \simeq R/xR$. Splitting is automatic.

Monomial Conjecture

Conjecture (Hochster '73)

If A is a local ring and $x_1, \ldots, x_d$ is a system of parameters for A, then for all $t \geqslant 0$,

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1}).$$

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- Equivalent to the Direct Summand Conjecture.
- True in equicharacteristic and $\dim(A) \leqslant 3$.

Strong Monomial Conjecture

Conjecture (SMC - Ranganathan '00)

Let A be a local ring with system of parameters $x_1, \ldots, x_d$. Let Q be a height one prime of A containing x_1. Then

$$x_1(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})Q \quad \textit{for all } t \geqslant 0.$$

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- Implies the Strong Direct Summand Conjecture.
- Can assume A is a complete local domain.

Setup:
$A =$ complete local ring
$S \twoheadrightarrow A$, where S is a complete intersection such that $\dim(S) = \dim(A)$.

Theorem (Strooker-Stückrad '93)

The Monomial Conjecture is equivalent to the following statement: Given any complete intersection S and any ideal of zerodivisors $I \subseteq S$, then $\mathrm{Ann}_S(I)$ is not contained in any parameter ideal.

Proposition (-)

Let $A, S, Q, \mathbf{x} = x_1, \ldots, x_d$ *be as in SMC. Set* $J = \mathrm{Im}(\mathrm{Hom}_S(Q, S)) \subseteq \mathrm{Hom}_S(S, S) = S$ *induced by* $S \to Q$ *sending* $1 \mapsto x_1$. *Then*

$$x_1(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})Q \quad \text{for all } t \geqslant 0.$$

if and only if

$$J \not\subseteq \mathbf{x}S.$$

A = complete local ring
S = complete intersection
$S \twoheadrightarrow A$
$\dim(S) = \dim(A)$
$\mathbf{x} = x_1, \ldots, x_d$ a s.o.p.
$Q \subseteq A$ with $\mathrm{ht}(Q) = 1$ and $x_1 \in Q$

Theorem (-)

The Strong Monomial Conjecture holds for all local rings if and only if it holds for all local almost complete intersection rings.

Theorem (-)

Let $A, Q, \mathbf{x} = x_1, \ldots, x_d$ be as in SMC. Suppose the residue field of A is infinite. Then there exist $y_1, \ldots, y_{d-1} \in A$ such that $(x_1, \ldots, x_d) = (y_1, \ldots, y_{d-1}, x_d)$ and such that "SMC holds for $y_1, \ldots, y_{d-1}, x_d^s$ for $s \gg 0$."

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- Proof uses Samuel's theory of superficial elements and the following two results:

A = complete local ring
S = complete intersection
$S \twoheadrightarrow A$
$\dim(S) = \dim(A)$
$\mathbf{x} = x_1, \ldots, x_d$ a s.o.p.
$Q \subsetneq A$ with $\mathrm{ht}(Q) = 1$ and $x_1 \in Q$

Proposition (Dutta)

Let $x_1, \ldots, x_d$ be a system of parameters of a local ring A such that

$$e_{(x_1,\ldots,x_d)}(A) = e_{(x_2,\ldots,x_d)}(A/(x_1)) = \cdots = e_{(x_d)}(A/(x_1, \ldots, x_{d-1})).$$

Then $H_i(x_1, \ldots, x_{d-1}; A)$ is a module of finite length for every $i > 0$.

Proposition (-)

Let $A, S, Q, \mathbf{x}$ be as in the setup and let $J = \mathrm{Im}(\mathrm{Hom}_S(Q, S))$. Then the Strong Monomial Conjecture holds for A if and only if $\ell(J/\mathbf{x}J) > \ell(H_1(\mathbf{x}; S/J))$.

A = complete local ring
S = complete intersection
$S \twoheadrightarrow A$
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